Pricing of Derivatives Contracts under Collateral Agreements: Liquidity and Funding Value Adjustments *

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1 Introduction

In current days most of contracts dealt in the interbank OTC derivatives are collateralized. A collateral agreements is characterized by the following features, amongst others:

- **Collateral, or Initial Margin**: is the amount of money (or other liquid assets) of that a counterparty has to post to the other when the derivatives contract has a negative NPV to the former.
- **Variation Margin**: is the variation of the collateral subsequent to a variation in the NPV of the derivatives contract.
- **Maintenance Margin**: is the level of the collateral below which it is not possible to drop after the variation margins are posted. If the balance drops below the level, the initial margin has to be restored.

The CSA is a contract whereby a percentage (typically 100%) of the negative NPV fully collateralized by the relavent counterparty, and a daily variation margin, equal to 100% of the daily variation of the NPV, is posted by the party which the variation was negative to. Under such an agreement, mantainance margin is redundant. The total collateral amount (initial + variations) is remunerated at a prespecified rate.

It should be noted also that CSA agreements usually operates on an aggregated base: the NPVs of all contracts (also for different types of underlying) included in a netting set

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are summed algebraically and the net amount is posted as collateral by the counterparty who has a negative total NPV. Also minimum transfer amount and threshold clauses applies. We will not dwell on the netting, minimum transfer amounts and thresholds in what follows.

Futures contracts have features similar to CSA agreements, **but**: the initial margin (collateral) is a determined as a small percent of the value of the future delivery (futures price times the notional of the contract). Variation margins occur daily but differently from the CSA, they can be withdrawn if positive to a counterparty, provided that the maintenance margin has not be eroded. In the end they are not real variations margins, but a daily liquidation of the variation of the terminal value of the contract. There is remuneration for the initial margin, no remuneration for the variation margins.

In what follows we try and analyse the pricing of derivatives under a CSA agreement, without considering netting, minimum transfer amounts and thresholds. So we will investigate the pricing of a contract on a "stand-alone" basis, although we are aware that "incremental" pricing, when netting is considered, may alter significantly the result and then it should not be overlooked if one wants to apply a more refined methodology.

Fujii and Takahashi [6] is a work closely relating to the analysis below: they study the effects of imperfect collateralization and they introduce a decomposition of the total contract's value which resembles the one we offer below, including also the bilateral CVA. On the other hand, we extend their analysis to include effects that funding costs have on the final contract's value, disregarding the residual counterparty credit risk due to imperfect collaterlazation.

Another recent work related to our analysis is in Pallavicini et al. [10]: they study the effects of partial collateralization on bilateral credit risk, taking into account also the costs due to different rates paid and received on the collateral account. Although their pricing fomulae somehow encompass also ours below, we think that we offer a different and intuitive approach to include funding costs, with the same remark as before that we do not consider credit risk. We also have to stress the fact that Pallavicini et al. [10] focus on deriving a general formula to calculate the **price** of the contract,[1] whereas we try and derive which is the **value** of the contract to a counterparty.

1.1 A Brief Digression on Price and Value

The difference between price and value has been investigated in economic theory, but economists (whether classical, or neoclassical, or Marxist) typically refer those terms to commodities. When a financial contract is not executed by simply (almost) immediately delivering an asset (in which case it can be assimilated to the purchase/sale of a commodity), but on the contrary it implies a given performance by possibly both parties for an extended duration, then price and value should be defined in a slightly more refined way.

We define price (from one of the parties' perspective) of a derivative contract the terms that both parties agree upon when closing the deal. These take into account the present value of expected profits and losses, considering all the costs and the losses due

[1] In fact they take into account the bilateral counterparty credit risk, and the cost of funding borne by each of the counterparties involved.

to counterparty credit risk, funding and liquidity premiums, for both parties. When both parties have an even bargaining power, they have to acknowledge the other party all the risks and the costs it bears, so that the final price includes the total (net) risks and costs borne by both parties.

On the other hand, we define production costs (to one of the parties) of a derivative contract the present value of the costs paid to attain the intermediate and final pay-off until the expiry, considering also the costs and the losses due to counterparty credit risk, funding and liquidity premiums, related to that specific party. Production costs, assuming no other margin is charged, is the also how much the contract is worth to the party, or alternatively said they are value of the contract.

For a counterparty the value of a contract can be exactly the price if it has enough bargaining power to completely transfer production costs (still excluding other extra-profit margins) to the counterparty, without recognizing in the setting of the contract's terms the costs and risks born by the other party. When the bargaining process involves counterparties with even bargaining power, then the value of the contract to each of them will be lower than the price as they are both yielding a share of the value to cover each other's risks and costs. The price and the value of a contract are the same also when both parties operate in a prefect and frictionless market, where there are no transaction costs and counterparty risks. In fact in this case they will agree on a production cost of the contract that is the same for both.

There are profound implications for the investment banking business from the definition above: when financial institutions of even bargaining power trade derivative contracts, they (both) are destroying their franchise since they are not able to fully transfer the total costs to the counterparty, being forced to accept a worsening of the terms of the contracts to acknowledge other party's costs. That means that they have to make up for the losses due to the difference between price and value with other counterparties that have less bargaining power, so as to restore to eroded franchise. So, weaker counterparties not only cannot heap on the stronger parties the remuneration for the risks and the costs born, but they will have also to pay for the costs charged by third parties to the party they are dealing with.

2 Pricing in a Simple Discrete Setting

Assume underlying asset S at time 0, and it can go up to $S_u = Su$ or down to $S_d = Sd$, with $d < 1$, $u > 1$ and $u \times d = 1$ in next period. Let V^C be the price of a contingent claim at time 0 (the "C" at the exponent stands for "collateralized"), and V_u^C and V_d^C its value when the underlying jumps to, respectively, to S_u and S_d. C is the value of the collateral account to be posted to the counterparty holding position in the contingent claim when the NPV is positive to it; the collateral account earns the collateral rate c. We will assume that a percent γ of the contract's NPV is continuously collateralized, so that at any time $C = \gamma V$.[2] B is the value at of a bank account earning at each period

[2]When $\gamma < 100\%$, that is there is not a full collateralization, then a residual counterparty credit risk should be priced into the contract. To isolate the effect of the collateral, we do not consider in

the risk-free rate r. In this framework, following the classical binomial approach by Cox and Rubinstein [4], we try and build a portfolio of underlying asset S and bank account B perfectly replicating the value of the contingent claim in each state of the world, jointly with the value of the collateral account. In other words, we want to replicate a long position in the collateralized contingent claim.

To do so, we have to set the following equalities in each of the two state of the world (*i.e.*: possible outcomes of the underlying asset's price):

$$V_u^C - C(1+c) = \alpha Su + \beta B(1+r) \tag{1}$$

and

$$V_d^C - C(1+c) = \alpha Sd + \beta B(1+r) \tag{2}$$

Equation (1) states that the value of the contingent claim V_u^C, when the underlying jumps to S_u from the starting value S, minus the value of the collateral account, must be equal to the value of the replicating portfolio, comprised of α units of the underlying and β units of the bank account. The collateral account at the end of the period will be equal to the initial value C at time 0, plus the interest rate accrued c. The replicating portfolio has to be revalued at the prices prevailing at the end of the period, that is S_u for the underlying asset and the initial value B plus accrued interest r for the bank account. In a very similar way, equation (2) states that the value of the contingent claim, minus the value of the collateral account, must be equal to the value of the replicating portfolio when the underlying jumps to S_d.

Equations (1) and (2) are a system that can be easily solved for quantities α and β, yielding:

$$\alpha = \Delta = \frac{V_u^C - V_d^C}{(u-d)S} \tag{3}$$

and

$$\beta = \frac{uV_d^C - dV_u^C - (1+c)C(u-d)}{(u-d)B(1+r)} \tag{4}$$

We have indicated $\alpha = \Delta$ because it is easily seen in (3) that it is the numerical first derivative of the price of the contingent claim with respect to the underlying asset, usually indicated so in the Option Pricing Theory.

If the replicating portfolio is able to mimic the pay-off of the collateralized contingent claim, then its value at time 0 is also the arbitrage-free price of the collateralized contingent claim:

$$V^C - C = \Delta S + \beta B = \frac{V_u^C - V_d^C}{(u-d)} + \frac{uV_d^C - dV_u^C - (1+c)C(u-d)}{(u-d)(1+r)} \tag{5}$$

It is possible to express (5) in terms of discounted expected value under the risk neutral measure and, recalling that $C = \gamma V^C$ and rearranging, we get:

$$V^C \frac{[(1+r)(1-\gamma) + (1+c)\gamma]}{1+r} = \frac{1}{1+r}[pV_u^C + (1-p)V_d^C] \tag{6}$$

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this work the counterparty credit risk still present in case of a imperfect collaterlaization. The inclusion of the counterparty credit risk in the pricing fo derivative contracts, considering also the funding costs due to the collateral management, has been studied in Pallavicini et al. [10], where probably the most comprehensive pricing formula is presented.

with $p = \frac{(1+r)-d}{u-d}$. The value of the collateralized contingent claim V^C is trivially:

$$V^C = \frac{1}{[(1+r)(1-\gamma)+(1+c)\gamma]}[pV_u^C + (1-p)V_d^C] \tag{7}$$

that is the expected risk neutral value multiplied by the factor $\frac{[(1+r)(1-\gamma)+(1+c)\gamma]}{1+r}$, which makes the final formula look like the expected value discounted with a rate which is a weighted average of the risk free and collateral rate, in stead of the risk-free rate only, albeit we still are in a risk-neutral world.

The right-hand side of the equation (6) is also equal to the expression one would get when trying to replicate a contingent claim without any collateral agreement.[3] Let V^{NC} be the value of such claim, then we have:

$$V^C \frac{[(1+r)(1-\gamma)+(1+c)\gamma]}{1+r} = V^C - \gamma\frac{r-c}{1+r}V^C = V^{NC} \tag{8}$$

Equation (8) states that the non collateralized contingent claim is equal to an otherwise identical collateralized claim, minus a quantity that we name Liquidity Value Adjustment (**LVA**) and precisely define as follows:

Definition 2.1. *The* **LVA** *is the discounted value of the difference between the risk-free rate and the collateral rate paid (or received) on the collateral, and it is the gain (or loss) produced by the liquidation of the NPV of the derivative contract due to the collateralization agreement.*

The fact that we are still working in a risk-neutral world is confirmed by the expected return on the underlying asset:

$$pS_u + (1-p)S_d = (1+r)S$$

that is equal to the risk-free rate.

It is worth mentioning the fact that by extending the binomial approach to a multi-period setting, thus introducing a dynamical replicating strategy whereby the contingent claim is replicated by dynamically re-balancing the underlying asset and bond portfolio, the final result of the replica is not the terminal pay-off of the contingent claim, but it includes both the latter and the terminal value of the cumulated losses/gains arising from the **LVA**. This has some very important implications at a dealing room level that we will try and examine in section 7.

The practical example below will clarify how the replication argument works under collateral and the pay-off attained at the expiry.

Example 2.1. *Assume*[4] *we want to price a call option fully collateralized* ($\gamma = 100\%$) *written on an underlying asset whose staring value is* 80*, which is also the strike price.*

[3]This can be easily verified by setting $\gamma = 0$ in equation (6).

[4]This example is the same as in the classical work by Cox, Ross and Rubinstein [4], with the inclusion of the collateral agreement not present there.

The risk free rate for one period is $r = 0.10$, whereas the collateral rate for each period is $c = 0.06$. The option expiries in three periods; at the end of each period the underlying asset can jump upward of downward by a factor, respectively, $u = 1.5$ and $d = 0.5$, so that the probability to have a jump up is $p = 0.6$. In the table 1 we show the evolution of the underlying asset price and the associated probability below each possible outcome.

			270 *0.216*
		180 *0.36*	
	120 *0.6*		90 *0.432*
80		60 *0.48*	
	40 *0.4*		30 *0.288*
		20 *0.16*	
			10 *0.064*

Table 1: Evolution of the underlying asset and associated probabilities below each possible outcome (in italics).

The value of the option can be computed via (7), by working out the backward recursion starting from the known terminal pay-off. The value of the option at each point of the binomial grid is also the value of the collateral account (with the reverse sign). Table 2 shows the result and we can read that the value of the collateralized option at time 0 is $V^C = 38.0851$.

			190
		111.3208	
	65.1477		10
38.0851		5.6604	
	3.2039		0
		0	
			0

Table 2: Value of the call option at each point of the grid, and of the collateral account (same with reverse sign).

The replicating portfolio can be built be computing the Δ for the underlying asset and the quantity β of bank account needed to finance the purchase. In table 3 the Δ is shown for each node of the binomial tree along a predefined path of the underlying asset (it is arbitrary and for illustration purposes only); below each Δ, we indicate also the quantity to trade in the bank account, plus the interests paid on the amount of bank account traded in the previous period. At the end of the last period we consider both types of jumps, so as to examine what happens when the option terminates in the money or out of the money.

		0.8805				1.0000
0.7743	↗	*-38.9892*	↘			*-75.0638*
-23.8586				0.1667	↗	
				-0.0580	↘	
						0.0000
						4.9362

Table 3: Amount of underlying asset to trade at each point of the predefined path. Below each Δ, the amount of bank account plus accrued interests from the previous period, are shown (in italics).

At time 0 *the quantity of underlying to hold in the portfolio, to replicate one call option, is* 0.7743. *To finance this purchase, we have to borrow money by selling a bank account for an amount of* -23.8586. *The difference is the amount of money we have to invest to start up the replication strategy, and it is exactly the value of the option at time* 0.

At time 1, $\Delta = 0.8805$ *so that we have to buy more asset and we have to increase the selling of bank account to borrow more money, besides paying the accrued interest on the initial borrowing of* 23.8586, *that we still have. The value of the bank account account is then* -38.9892. *When we arrive at the last period either with one asset in the portfolio, and a bank account value of* -75.0638, *when the options expiries in the money; otherwise we end up with no asset and a bank account value of* 4.9362 *when the option expiries out of the money.*

There is an additional amount of money to be borrowed when replicating the collateralized option, and this is the amount needed to finance the collateral account value. Hence, a long position in a collateralized option entails a short position in the collateral account, since we have a cash amount of money equal to the value of the contingent claim. The total cost to replicate the collateral account is given by the difference between the risk-free and collateral rate, times the amount of the collateral account at the previous period. In table 4 we show the cost associated to each point of the predefined path we have chosen for the underlying asset; the cost is nil a time 0 and has to be financed for the other periods.

		1.5234				0.2264
	↗		↘		↗	
0				2.6059		
					↘	
						0.2264

Table 4: Cost to replicate the collateral account at each point of the predefined underlying asset's path.

Let us investigate now which is the replicated value of the call option. This is shown in table 5, where we re-valuate at each point of the predefined path the replicating portfolio as far as the quantity of underlying asset and the bank account needed to finance its purchase are concerned. As it can be easily seen, the replicating portfolio is not exactly mimicking the value of the call, and actually at the expiry the two possible pay-offs (i.e.: 10 *when the call terminates in the money and* 0 *otherwise) are not matched in both cases.*

		66.6711	$\searrow$		$\nearrow$ 14.9362
38.0851	$\nearrow$			9.9420	
					$\searrow$ 4.9362

Table 5: Replica of the call option with the underlying asset and bank account portfolio.

The error in the replica is exactly equal to the cost to finance the collateral account. Actually, when adding the sum of values from table 4, and we compound them at each period with the risk-free rate, we get the total result in table 6, that shows that the at each period, including at the expiry, the call option value is exactly replicated. At the first period, the total replica is 66.6711 *plus the cost of the collateral account* 1.5234*, for a total of* 65.14774*, which is exactly the call value in table 2. At the end of the second period, we need to compound* 1.5234 *at the risk-free rate (*0.10*), and sum it to the cot for the second period (*2.6059*). By adding this total cost to the replicated value of the option (*9.9420*) we finally get the total replication value of* 5.6604*, once again the same as in table 2. By the same token we can derive also the total replication value at the expiry for the two cases of moneyness.*

65.1477	$\searrow$			10
		5.6604	$\nearrow$	
			$\searrow$	
				0.0000

Table 6: Call replica including the cost to finance the collateral account.

3 Replicating Portfolio in Continuous Time

Now we extend the binomial approach we sketched above in a continuous and more general setting. Assume the underlying asset follows a dynamics of the type:

$$dS_t = (\mu_t - y_t)S_t dt + \sigma_t S_t dZ_t \tag{9}$$

The underlying has a continuous yield of y_t and a volatility σ_t.

The contingent claim dynamics is derived via the Ito's lemma:

$$dV_t = \mathcal{L}^{\mu} V_t + \sigma_t S_t \frac{\partial V_t}{\partial S_t} dZ_t \tag{10}$$

where we used the operator $\mathcal{L}^a \cdot$ defined as:

$$\mathcal{L}^a \cdot = \frac{\partial \cdot}{\partial t} + a_t S_t \frac{\partial \cdot}{\partial S_t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 \cdot}{\partial S_t^2} \tag{11}$$

Besides, we will set also $\Delta_t = \frac{\partial V_t}{\partial S_t}$ in what follows. The cash collateral account dynamics is defined as

$$dC_t = \gamma dV_t + c_t C_t dt \tag{12}$$

c_t collateral rate, r_t is the funding/investment rate and

$$\begin{aligned} C_0 &= \gamma V_0 \\ C_{T^-} &= E^Q\left[\int_0^{T^-} e^{-\int_u^{T^-} c_v dv} \gamma dV_u\right] \\ C_T &= 0 \end{aligned}$$

Collateral can be seen as a bank account (actually, it is a bank account), so that receiving cash collateral means being short the collateral account (as when one shorts a bond and receives cash). At the end the collateral is returned to the transferor (at the same time the final pay-off of the contingent claim is received by the transferee).

The bank cash account evolution is deterministic and equal to:

$$dB_t = r_t B_t dt \tag{13}$$

Also in this case, as above for the cash collateral account, being short B means receiving cash.

At time 0, the replication portfolio in a long position in the derivatives V, cash-collateralized, is set up. It comprises a given quantity of the underlying asset and of bank account such that their value equals the starting value of the contract and of the collateral account:

$$V_0 - C_0 = \alpha_0 S_0 + \beta_0 B_0 \tag{14}$$

We have to find a trading strategy $\{\alpha_t, \beta_t\}$, such that it satisfies the following well-known conditions:

1. Self financing condition, that is: no other investment is required in operating the strategy besides the initial one:

$$\begin{aligned} \alpha_t S_t + \beta_t B_t =& \alpha_0 S_0 + \beta_0 B_0 \\ &+ \int_0^t \alpha_u(\mu_u - y_u) S_u du + \int_0^t \alpha_u \sigma_u S_u dZ_u + \int_0^t \beta_u dB_u + \int_0^t \alpha_u y_u S_u du \end{aligned} \tag{15}$$

2. Replicating condition, that is: at any time t the replicating portfolio's value equals the value of the contract and of the collateral account:

$$V_t - C_t = \alpha_t S_t + \beta_t B_t \tag{16}$$

for $t \in [0, T]$.

We can write the evolution of the replicating portfolio as:

$$\alpha_t dS_t + \beta_t dB_t = \alpha_t(\mu_t - y_t)S_t dt + \alpha_t \sigma_t S_t dZ_t + \beta_t r_t B_t dt + \alpha_t y_t S_t \tag{17}$$

On the other hand:

$$dV_t - dC_t = \mathcal{L}^{\mu-y} V_t dt + \sigma_t S_t \Delta_u dZ_t - \gamma dV_t - c_t C_t dt \tag{18}$$

Equating (17) and (18) and imposing the self-financing and replicating conditions, we get:

$$\begin{aligned}&\mathcal{L}^{\mu-y} V_t dt + \sigma_t S_t \Delta_u dZ_t - \gamma dV_t - c_t C_t dt = \\ &\alpha_t(\mu_u - y_t)S_t dt + \alpha_t \sigma_t S_t dZ_t + \beta_t r_t B_t dt + \alpha_t y_t S_t\end{aligned} \tag{19}$$

We can determine the α and β such that the stochastic part in (19) is cancelled out:

$$\alpha_t = \Delta_t \tag{20}$$

$$\beta_t = \frac{V_t - C_t - \Delta_t S_t}{B_t} \tag{21}$$

Substituting in (19):

$$\mathcal{L}^{r-y} V_t dt = r_t V_t dt + \gamma dV_t - (r_t - c_t) C_t dt \tag{22}$$

Let us split (22) in two parts. The first is the standard PDE under the risk neutral argument:

$$\mathcal{L}^{(r-y)} V_t = r_t V_t \tag{23}$$

The second part is more unusual:

$$\gamma\big(\mathcal{L}^{\mu} V_t dt + \sigma_u S_u \Delta_u dZ_u\big) + c_t C_t dt = r_t C_t dt \tag{24}$$

and it is the evolution of the collateral account, in the **real** world measure, equating the cost of the bank account used to finance it.

Equation (22) has a solution that can be found by means of the Feynman-Kac theorem:

$$V_0^C = -C_0 + E^Q\left[e^{-\int_0^T r_u du} V_T + \int_0^T e^{-\int_0^u r_v dv}(r_u - c_u) C_u du - \int_0^T e^{-\int_0^u r_v dv} \gamma dV_u\right] \tag{25}$$

Considering again the fact that the collateral at expiry will be paid back to the counterparty who posted it, $C_T = 0$, we have:

$$E^Q\left[\int_0^T e^{-\int_0^u r_v dv} \gamma dV_u\right] = E^Q\left[\int_0^{T^-} e^{-\int_0^u r_v dv} \gamma dV_u - e^{-\int_0^T r_v dv} \gamma V_T\right] = \gamma V_0 = -C_0$$

so that equation (25) can be written as:

$$V_0^C = E^Q\left[e^{-\int_0^T r_u du} V_T\right] + E^Q\left[\int_0^T e^{-\int_0^u r_v dv}(r_u - c_u)C_u du\right] \tag{26}$$

Equation (28) states the same result we have derived in a binomial setting above, that is: a collateralized claim is equal to value of an otherwise identical non-collateralized claim, minus the present value of the cost incurred to finance the collateral, or the **LVA**:

$$V_0^C = V_0^{NC} + \mathbf{LVA}$$

It is worth mentioning the fact that we still have not introduced any credit risk until now, so that the **LVA** cannot be confused with any adjustment due to the risk of default. On the other hand, it is still possible to derive an arbitrage free price when risk-free rate and collateral rate are different, something counterintuitive at first sight.

Recalling that $C_t = \gamma V_t$, equation (22) can be equivalently decomposed as:

$$\mathcal{L}^{(r-y)} V_t dt = [r_t(1-\gamma) + c_t\gamma]V_t dt + \gamma dV_t \tag{27}$$

The solution to (27) applying the Feynman-Kac theorem is:

$$V_0^C = E^Q\left[e^{-\int_0^T [r_u(1-\gamma)+c_u\gamma]du} V_T\right] - C_0 - E^Q\left[\int_0^T e^{-\int_0^u [r_u(1-\gamma)+c_u\gamma]dv} dV_u\right] \tag{28}$$

The second part on right-hand side is nil, since as before:

$$\begin{aligned} E^Q\left[\int_0^T e^{-\int_0^u [r_u(1-\gamma)+c_u\gamma]dv}\gamma dV_u\right] &= E^Q\left[\int_0^{T-} e^{-\int_0^u [r_u(1-\gamma)+c_u\gamma]dv}\gamma dV_u - e^{-\int_0^T [r_u(1-\gamma)+c_u\gamma]dv}\gamma V_T\right] \\ &= \gamma V_0 = -C_0 \end{aligned}$$

So:

$$V_0^C = E^Q\left[e^{-\int_0^T [r_u(1-\gamma)+c_u\gamma]du} V_T(S^{r-y})\right] \tag{29}$$

We have added the dependency of the value of the claim on the underlying price, whose drift is indicates in the superscript. Thus we have a perfect analogy with the discrete case we examined above.

When the deal is fully collateralized (*i.e.*: $\gamma = 100\%$), the discount rate in Equation (29) collapses to the collateral rate c_t, and this is a a well known result (see, amongst others, Fujii et al. [5], Mercurio [7] and Piterbarg [11]). We think that equation (28) offers more insight. Actually, discounting with the collateral rate is a way to use an effective rate producing the effects of the risk-free discounting and of the **LVA**. Nevertheless, if one wishes to disentangle the effects then she should resort to (29). As an example, in a dealing room the correct evaluation of the **LVA** allows to correctly allocate the liquidity costs related to the collateralization on the relevant desks. If a collateral desk exists, the **LVA** can be the compensation it receives to manage a given deal, whereas the trading desk closing the deal will be left with just the risk-free value of the contract that has to be managed.

4 Pricing with Funding Rate Different from Investment Rate

Assume that the operator of the replication strategy is a bank. The difference between the investment and funding rate is due mainly to credit factors (barring the trivial bid/ask factor and liquidity premiums), so that when considering rates actually paid or received by the bank, we should model also the default event. Nevertheless this is not necessary since we are assuming that the pricing is operated from the bank's perspective.

Actually, the funding rate r^F that a bank has to pay, when financing its activity, should be considered just a cost from its perspective, on the base of *the on-going concern principle*. On the other hand, from the lender perspective, the spread over the risk-free rate paid by the bank, is the remuneration for bearing the risk of default of the borrowing bank (see Castagna [2] for a detailed discussion on this. For an alternative view, see Morini and Prampolini [9]).

When the bank sells a bank account, then it will pay the interest r^F on the received funds until the maturity; conversely, when buying a bank account, we assume there is a risk-free borrower which pays the risk-free rate r. The evolution of the bank account in (13) becomes:

$$dB_t = \widetilde{r}_t B_t dt \tag{30}$$

where $\widetilde{r}_t = r_t \mathbf{1}_{\{\beta>0\}} + r_t^F \mathbf{1}_{\{\beta<0\}}$ and $\mathbf{1}_{\{\}}$ is the indicator function equal to 1 when the condition at the subscript is verified. If the quantity β of the bank account is negative (*i.e.*: the bank borrows money) then the bank account grows at the funding rate r_t^F; when the quantity β is positive (*i.e.*: the bank lends money) than the bank account grows at the risk-free rate r_t. If a risk-free borrower does not exist, so that we actually have to buy bank accounts issued by other defaultable banks, then we can invest at a rate $r^B > r$, and the difference between the two rates is the remuneration for the credit risk. The expected return earned on the investment will be in any case the risk-free rate r. The default of the counterparty, whom the bank lends money to, will anyway affect the performance of the replication strategy of the contingent claim, so that the counterparty credit risk should be eliminated or mitigated whenever this is possible. We will come back on this issue later on.

Assuming that the funding rate is the risk-free rate plus a spread s_t^F, we can write the rate at which bank account's interests accrues as:

$$\widetilde{r}_t = r_t + s_t^F \mathbf{1}_{\{\beta<0\}} \tag{31}$$

Replacing the risk-free rate r_t with $\widetilde{r}_t$ in equation (22), one gets:

$$\mathcal{L}^{\widetilde{r}-y} V_t dt = \widetilde{r}_t V_t dt + \gamma dV_t - (\widetilde{r}_t - c_t) C_t dt \tag{32}$$

From (32) we can easily derive the two ways to express the contingent claim's value at time 0 equivalent to formulae (28) and (29), respectively as:

$$V_0^C = E^Q\left[e^{-\int_0^T \widetilde{r}_u du} V_T\right] + E^Q\left[\int_0^T e^{-\int_0^u \widetilde{r}_v dv} (\widetilde{r}_u - c_u) C_u du\right] \tag{33}$$

and

$$V_0^C = E^Q\left[e^{-\int_0^T [\widetilde{r}_u(1-\gamma)+c_u\gamma]du} V_T\right] \tag{34}$$

Equation (33) offers the decomposition of the collateralized contract value as the sum of the otherwise identical non-collateralized deal and of the **LVA**.

To get even more insight and to allow for a further decomposition that can be useful to allocate revenues and costs within a dealing room, we rewrite equation (32) as:

$$\mathcal{L}^{r-y}V_t dt = r_t V_t dt + \gamma dV_t - (r_t - c_t)C_t dt + s_t^F \mathbf{1}_{\{\beta<0\}}(V_t - C_t - \Delta_t S_t)dt \tag{35}$$

The solution to (35) is:

$$V_0^C = V^{NC} + \mathbf{LVA} + \mathbf{FVA} \tag{36}$$

where V^{NC} is the price of the non-collateralized contract assuming no funding spread, the **LVA** is the liquidity value adjustment originated by the difference between the collateral and risk-free rate:

$$\mathbf{LVA} = E^Q\left[\int_0^T e^{-\int_0^u r_v dv}(r_u - c_u)C_u du\right] \tag{37}$$

and finally **FVA** is the funding value adjustment due to the funding spread and paid to replicate the contract and the collateral account:

$$\mathbf{FVA} = E^Q\left[-\int_0^T e^{-\int_0^u r_v dv} s_u^F \mathbf{1}_{\{\beta<0\}}(V_u - C_u - \Delta_u S_u)du\right] \tag{38}$$

where β has been defined above. The **FVA** is the correction to the risk-free value of the non-collateralized contract that has to be (algebraically) added to the **LVA** correction. We define it as:

Definition 4.1. *The* **FVA** *is the discounted value of the spread paid by the bank over the risk-free interest to finance the net amount of cash needed for the collateral account and the underlying asset position in the dynamic replication strategy.*

It is interesting to untangle the total **FVA** in its components: this decomposition is not essential as far as the pricing is concerned, but it is very useful within a dealing room to charge the desks involved in the trading (we will dwell more on this later on). We then isolate a first part of the total **FVA** due to the funding cost of the premium's and the collateral's replication strategy:

$$\mathbf{FVA^P} = E^Q\left[-\int_0^T e^{-\int_0^u r_v dv} s_u^F \mathbf{1}_{\{\beta<0\}}(V_u - C_u)du\right] \tag{39}$$

and a second part referring to the funding cost born to carry the position in the underlying asset in the replication strategy:

$$\mathbf{FVA^U} = E^Q\left[\int_0^T e^{-\int_0^u r_v dv} s_u^F \mathbf{1}_{\{\beta<0\}}\Delta_u S_u du\right] \tag{40}$$

Hence the total funding value adjustment is $\mathbf{FVA} = \mathbf{FVA^P} + \mathbf{FVA^U}$. Since in both components the indicator function $\mathbf{1}_{\{\beta<0\}}$ appears, the **FVA** of the single components takes into account that, at the financial institution's level, the net funding need is considered, thus single trading desks enjoy also a funding benefit at an aggregated level. As an example, consider the **FVA** for the cost born to fund the underlying asset's position: the derivatives desk should pay the funding costs when it has a positive position, but this cost is paid only if the net bank account's amount is negative ($\beta < 0$). When the underlying asset's position is positive but the net amount in the bank account is positive ($\beta > 0$), the derivatives desk will not be charged by any funding cost, although it actually requires funds to buy the asset.

We can now analyse five different cases:

1. Assume one has to replicate a contingent claim with a constant positive sign NPV (*e.g.*: a long European call option) with a constant positive sign Δ_t. Since $V_t - C_t - \Delta S_t$ is always negative (implying borrowing), the total amount of the bank account β is always negative, implying that at any time we have to borrow money in the replica at the rate r^F. The pricing equation (35) reads then:

$$\mathcal{L}^{r-y} V_t dt = r_t V_t dt + \gamma dV_t - (r_t - c_t) C_t dt + s_t^F (V_t - C_t - \Delta_t S_t) dt \tag{41}$$

   Although the decomposition in (36) still applies, the pricing can be performed very simply by means of an effective discount rate:

$$V_0^C = E^Q \left[e^{-\int_0^T [\widetilde{r}_u^F (1-\gamma) + c_u \gamma] du} V_T(S^{r^F - y}) \right] \tag{42}$$

   So we can simply replace the risk-free rate with the funding rate paid by the bank and perform the same pricing as in the case when lending and borrowing rates are the equal.

   Equation (42) is a very convenient way to compute the price in 0 of the contracts, but it does not help allocating its components to the different desks of the bank.

2. When the same contingent claim (constant positive NPV and Δ) as in the point above is sold, the underlying asset has to be sold as well in the replication strategy, which implies that $\beta > 0$ and that the bank has to invest at the risk-free rate at any time. The pricing formula will the same as in formula (28) (with reversed signs since we are selling the contract). In this case the **FVA** will be nil. An example of this claim is a short European call option.

3. Assume now that the contingent claim has a constant positive sign NPV, but its replication implies a negative position in the underlying asset (*e.g.*: a long European put option), then we have again that $\beta > 0$ at any time. The pricing formula is also in this case (28), the same as in the case with no funding spread.

4. If the NPV has a constant negative sign and the replica entails a long position in the underlying (*e.g.*: short European put option), then the total amount of the bank account β is always negative, implying that at any time we have to borrow money in the replica at the rate r^F. The pricing formula will be the same as (42) in the first case above.

5. Finally, if the NPV has a constant positive or negative sign and the Δ can flip from one sign to the other, then it is not possible to determine the sign of the bank account amount β throughout the entire life of the contract. In this case the pricing formula (35) cannot be reduced to a convenient representation as in the cases above, and it has to be very likely computed numerically. Examples of contracts with non-constant sign Δ are exotic options, such as reverse knock-out.

S	100	σ	20%
K	100	r	2%
T	1	y	1%
c	2.5%	r^F	3%

Table 7: Input data for a European Call Option

From the analysis above it is also clear that when the contract is fully collaterlized, the effective discount rate is just the collateral rate, whereas the drift rate of the asset can be either the risk-free rate or the funding rate depending on whether the bank account preserves, respectively, always a positive or a negative sign during until the expiry.

Example 4.1. *We here show a simple example of how the ideas illustrated above can be applied in practice for a European call option on an underlying asset that can be an equity, an FX spot rate or a commodity. Typically the model used to price options in these cases is the standard Black&Scholes' one:*

$$\mathbf{C}(S, K, T, \sigma, r, y, d) = e^{-dT}[FN(d_1) - KN(d_2)] \tag{43}$$

where $N()$ is the Normal cumulated distribution function, $F = Se^{(r-y)T}$ is the forward price and:

$$d_1 = \frac{\ln \frac{F}{K} + 0.5\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Equation (43) valuates a call expiring in T, struck at K, when the underlying spot price is S.

Assume we want to price the call option with the input data in table 7. Since a European call option is a contract of the type in point 1 of the list we have shown above, the decomposition of the total value in the several components can be operated avoiding the computation of the integral entering in the definition of the **LVA** *and of the* **FVA**.

Actually, the risk-free, with a risk-free rate drift to set the forward price, non collateralized value of the call can immediately be computed as:

$$V^{NC} = V^{NC-RF-RD} = \mathbf{C}(S, K, T, \sigma, r, y, r)$$

The total adjustment of a collateralized option, considering funding costs bot into the discounting and into the drift of the asset to set the forward price, is:

$$\mathbf{TA} = V^{C-FU-FD} - V^{NC-RF-RD} = \mathbf{C}(S, K, T, \sigma, r^F, y, (r^F(1-\gamma)+c\gamma)) - \mathbf{C}(S, K, T, \sigma, r, y, r)$$

In the superscripts C/NC stands for collateralized/non collateralized, RF/FU stands for risk-free/funding rate discounting and RD/FD stands for risk-free/ funding rate drift.

The quantity **TA** *can be decomposed as follows:*

$$\begin{aligned}\mathbf{TA} =& V^{C-FU-FD} - V^{C-FU-RD} \\ &+ V^{C-FU-RD} - V^{C-RF-RD} \\ &+ V^{C-RF-RD} - V^{NC-RF-RD}\end{aligned}$$

Now, the **LVA** *is the third line of the equation above, and it can be computed by the Black&Scholes formula:*

$$\begin{aligned}\mathbf{LVA} &= V^{C-RF-RD} - V^{NC-RF-RD} \\ &= \mathbf{C}(S,K,T,\sigma,r,y,(r(1-\gamma)+c\gamma)) - \mathbf{C}(S,K,T,\sigma,r,y,r)\end{aligned}$$

The total **FVA** *is in the first two lines of the equation above. Namely, the difference between the collateralized option, discounted with the funding rate and the drift equal to the funding rate, and non-collateralized option, discounted with the risk-free rate and the drift equal to the risk-free rare:*

$$\mathbf{FVA} = V^{C-FU-FD} - V^{C-RF-RD}$$

We can decompose the total **FVA** *by recognizing that* $\mathbf{FVA}^{\boldsymbol{U}}$ *(i.e.: the funding value adjustment due to the underlying asset) is the difference in the first line of* **TA***:*

$$\begin{aligned}\mathbf{FVA}^{\boldsymbol{U}} &= V^{C-FU-FD} - V^{C-FU-RD} \\ &= \mathbf{C}(S,K,T,\sigma,r^F,y,(r^F(1-\gamma)+c\gamma)) - \mathbf{C}(S,K,T,\sigma,r,y,(r^F(1-\gamma)+c\gamma))\end{aligned}$$

The funding value adjustment due to the premium and collateral is :

$$\begin{aligned}\mathbf{FVA}^{\boldsymbol{P}} &= V^{C-FU-RD} - V^{C-RF-RD} \\ &= \mathbf{C}(S,K,T,\sigma,r,y,(r^F(1-\gamma)+c\gamma)) - \mathbf{C}(S,K,T,\sigma,r,y,(r(1-\gamma)+c\gamma))\end{aligned}$$

In table 8 we show the decomposition of the total option value into the components examined above, for different percentage γ of collateralization of the contract's NPV. It is quite obvious that for the non-collateralized contract ($\gamma = 0\%$), the **LVA** *is nil. It should be also noticed that the total values can be computed strightforwardly via formula (42), clearly obtaining the same result. Nevertheless with this slighlty longer procedure we are able to exactly disentangle the different cost's contributions.*

	γ: 100%		50%		0%	
V^{NC}	8.34941		8.34941		8.34941	
LVA	-0.04164		-0.02085		0.00000	
FVA	0.56381		0.52086		0.47792	
$\mathbf{FVA^P}$		*0.00000*		*-0.04154*		*-0.08308*
$\mathbf{FVA^U}$		*0.56381*		*0.56240*		*0.56099*
Total	8.87157		8.84942		8.82732	

Table 8: Decomposition of the call option's value into the risk-free, **LVA** and **FVA** components.

Example 4.2. *Assume now we have the same data as in the example 4.1 and that the European call is no more plain vanilla, but it has a barrier set above the strike level at* 135*. The option is and Up&Out call and it can be priced in a closed form formula in* *a Black&Scholes economy (see Castagna [1] for a thorough discussion of barrier options and for pricing formulae, with a focus on the FX market).*

In this case it is not possible to use the decomposition we have used in the example 4.1 because the Δ *of the Up&Out call can flip from one sign to the other, depending on the level of the underlying asset. We are in the fifth case of those listed above. In figure 1 we depict the* Δ *as a function of the underlying asset's price, for three times to maturity, progressively approaching the contract's expiry: the plots simply show what we have said. In this case we have to resort to a numerical integration of the formulae* (37) *and* (38).[5]

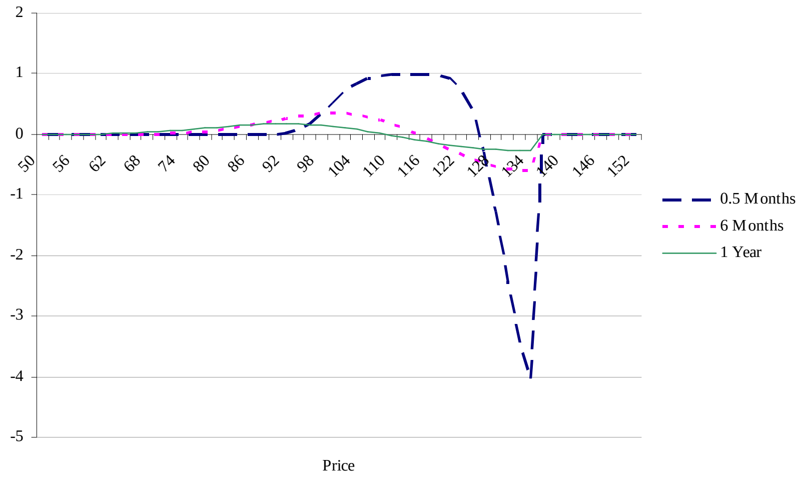

Figure 1: Delta of an Up&Out call option with different times to maturities, as a function of the underlying asset's price. The barrier is at 135 and all other data are the same as in example 4.1.

We present the decomposition of the price in table 9, only for the case when the contract if fully collateralized ($\gamma = 100\%$)*. This means that the* **FVA** *contains only the component referring to the underlying asset financing. It is quite easy to justify the lower amount of both the* **LVA** *and the* **FVA** *with respect to the corresponding European plain vanilla examined above.*

V^{NC}	4.04127
LVA	-0.02215
FVA	0.21679
Total	4.23502

Table 9: Decomposition of the value of an Up&Out call option in its non collateralized risk-free value, the **LVA** and the **FVA**

[5]We have used 45 time steps within the contract's duration of 1 year, and a 50 points Gauss-Legendre quadrature scheme for each time step.

5 Funding Rate Different from Investment Rate and Repo Rate

We now introduce the possibility to lend and borrow money (or, alternatively, the underlying asset) via a repo transaction. This is actually the way traders finance the buying of the underlying asset (typically in the stock market), by borrowing money and lending the asset as a collateral until the expiry of the contract.

A repo transaction can be seen as a collateralized loan, and the rate paid is lower than the unsecured funding rate of the bank, since in case of default of the borrower, the asset can be sold to guarantee the (possibly only partial) recovery of the lent sum. The difference between the repo rate r^E and the risk-free rate is due to the fact that the underlying asset can be worth less than the lent amount when default occurs: so the volatility of the asset and the probability of default both affect the repo rate.

We assume we have repo rate is the same when either borrowing money or lending money against the underlying asset (repo and reverse repo). This means that we are assuming that the two banks involved in the transaction have the same probability of default with the same recovery rate in the event of default. We will investigate the replication costs and the pricing formulae for four possible cases, as above.

Repo transaction should be the proper way to finance the buying of the underlying asset in the replication strategy. On the other hand, if we really want to consider the actual alternatives that a trader has to invest received sums in the less credit-risky way, reverse repo seems an effective option is most of cases. So, we go back to the case when there is no asymmetry between investment (lending) and funding rate, although the risk free rate is replaced by the repo rate. The amount to be lent/borrowed via the bank account is now:

$$\beta_t = \frac{V_t - C_t}{B_t} \tag{44}$$

whereas the quantity $\alpha_t = \Delta_t$ of underlying asset is repoed/reverse repoed, thus paying/receiving the interest $r_t^E \Delta_t S_t$. Replacing these quantities in equation (22), one gets:

$$\mathcal{L}^{r^E - y} V_t dt = \widetilde{r}_t V_t dt + \gamma dV_t - (\widetilde{r}_t - c_t) C_t dt \tag{45}$$

The solution to (45) is:

$$V_0^C = V^{NC} + \mathbf{LVA} + \mathbf{FVA} \tag{46}$$

where, as usual, V^{NC} is the price of the non-collateralized contract assuming no funding spread and repo, the **LVA** is the liquidity value adjustment due to the collateral agreement;

$$\mathbf{LVA} = E^Q \left[\int_0^T e^{-\int_0^u r_v dv} (r_u - c_u) C_u du \right]$$

and **FVA** is the funding value adjustment:

$$\mathbf{FVA} = E^Q \left[-\int_0^T e^{-\int_0^u r_v dv} [s_u^F \mathbf{1}_{\{\beta<0\}} (V_u - C_u) - s_t^E \Delta_u S_u] du \right] \tag{47}$$

The **FVA** is in this case split in the funding cost needed to finance the collateral ($s_u^F \mathbf{1}_{\{\beta<0\}} (V_u - C_u)$) and the spread of repo rate over the risk-free rate ($s_t^E = r_t^E - r_t$) paid on the position of amount Δ_t of the underlying asset.

To better understand how the total **FVA** is built up, we split formula (47) in two components: the first one is $\mathbf{FVA^P}$, the cost borne to fund the premium and the collateral and it is the same as in (39). The second part refers to the repo cost to buy or to sell the underlying asset to replicate the pay-off:

$$\mathbf{FVA^R} = E^Q\left[\int_0^T e^{-\int_0^u r_v dv} s_t^E \Delta_u S_u du\right] \tag{48}$$

Also in this case it is possible to re-write (46) in a more convenient fashion for computational purposes:

$$V_0^C = E^Q\left[e^{-\int_0^T [\tilde{r}_u^F(1-\gamma)+c_u\gamma]du} V_T(S^{r^E-y})\right] \tag{49}$$

Formula (49) applies in the five cases analysed in the previous section: the discount factor depends on the sign of the bank account needed to fund the collateral account, whereas the drift of the underlying asset is any case the repo rate r^E.

Example 5.1. *We revert to the example 4.1 above on the pricing of a European call option, and we now assume that the bank can buy or sell the underlying asset via repo transactions. We ascertain how the components of the total value change in this case. We still use the same inputs as in the table 7, and we add to them the repo rate set at* $r^E = 2.25\%$*, which is lower than the unsecured funding rate* $r^F = 3\%$*, but higher than the risk-free rate* $r = 2\%$ *to account for the volatility of the collateral (the underlying asset) and the possibility of a smaller collateral value on the default of the borrower (the bank).*

We can exploit once again the fact that the European option is a type of contract of the first case we analysed above, and make the same consideration we made in the example in section 4.1, and we define the **LVA** *as above:*

$$\begin{aligned}\mathbf{LVA} &= V^{C-RF-RD} - V^{NC-RF-RD}\\ &= \mathbf{C}(S,K,T,\sigma,r,y,(r(1-\gamma)+c\gamma)) - \mathbf{C}(S,K,T,\sigma,r,y,r)\end{aligned}$$

and the two components of the **FVA** *as:*

$$\begin{aligned}\mathbf{FVA}^{\boldsymbol{P}} &= V^{C-FU-RD} - V^{C-RF-RD}\\ &= \mathbf{C}(S,K,T,\sigma,r,y,(r^F(1-\gamma)+c\gamma)) - \mathbf{C}(S,K,T,\sigma,r,y,(r(1-\gamma)+c\gamma))\end{aligned}$$

$$\begin{aligned}\mathbf{FVA}^{\boldsymbol{R}} &= V^{C-FU-FD} - V^{C-FU-RD}\\ &= \mathbf{C}(S,K,T,\sigma,r^E,y,(r^F(1-\gamma)+c\gamma)) - \mathbf{C}(S,K,T,\sigma,r,y,(r^F(1-\gamma)+c\gamma))\end{aligned}$$

In table 10 we show the decomposition of the total option value into the different components, for different levels of percentage of collateralization.

6 Interest Rate Derivatives

When the pricing comes to interest rate derivatives, we have to consider the credit issue as a critical one. We have analysed the replication of a contingent contract with repo

	γ 100%		50%		0%	
V^{NC}	8.34941		8.34941		8.34941	
LVA	-0.04164		-0.02085		0.00000	
FVA	0.13860		0.09672		0.05483	
$\mathbf{FVA^P}$		*0.00000*		*-0.04154*		*-0.08308*
$\mathbf{FVA^R}$		*0.58601*		*0.13826*		*0.13791*
Total	8.44636		8.42528		8.40424	

Table 10: Decomposition of the call option's value into the risk-free, **LVA** and **FVA** components when underlying asset is traded via repo contracts.

transactions, which help virtually eliminating the credit risk, or at least making it negligible. Unfortunately it is not possible to replicate interest rate derivatives with such a low level of credit risk, since the replication strategy involves unsecured lending (besides the borrowing) as a part of the underlying itself. As an example, without credit risk, a FRA can be replicated by selling/buying a shorter maturity bond and buying/selling a longer maturity bond. With credit risk this strategy is clearly flawed since the counterparty whom we lent money to can go defaulted before the expiry of the bond.

This means that in practice basic interest rate derivatives are no more real derivatives, but primary securities that cannot be replicated by means of other primary securities (*e.g.*: bonds). The derivative contract can be made credit-risk free by a collateral agreement, but we cannot any more set up a strategy to replicate the pay-off and the evolution of the collateral account, as we have done above for derivatives on different assets. To illustrate the implications of the impossibility to implement a replications strategy, we analyse two contracts in what follows, a Forward Rate Agreement (**FRA**) and an Interest Rate Swap (**IRS**).

6.1 Forward Rate Agreement

Let us introduce the set up to price interest rates derivatives under collateral agreements.[6] Consider times t, T_{i-1} and T_i, $t \leq T_{i-1} < T_i$. The time-t forward rate is defined as the rate to be exchanged at time T_i for the LIBOR rate $L_i(T_{i-1}) = L(T_{i-1}, T_i)$ fixed at time T_{i-1}, in a $\mathbf{FRA}(t; T_{i-1}, T_i)$ contract, so that the contract has zero value at time t.

In the absence of credit risk (i.e.: in a single curve environment), the forward rate can be determined via a portfolio of long and short zero coupon bonds. Absence of arbitrage implies also the existence of a single, risk-free, discounting curve. Assume we have the discount curve denoted by D; we then have:

$$L^D(t; T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left[\frac{P^D(t, T_{i-1})}{P^D(t, T_i)} - 1 \right] \tag{50}$$

[6] The set-up and the notation is the same as in Mercurio [7].

The **FRA** fair forward rate can be set according to the definition of the contract:

$$\mathbf{FRA}(T_1;T_1,T_2) = \frac{T_i - T_{i-1}}{1 + L_i(T_{i-1})(T_i - T_{i-1})}\Big[L_i(T_{i-1}) - K\Big] \tag{51}$$

Assume now we are in a credit-risky economy. Selling and buying bonds do not allow to replicate the **FRA** pay-off since it is always possible that the counterparty whom we lent money to goes defaulted. The forward trading in the market, in this case should be considered simply as the expected value of the Libor at the fixing time. If we accept the fact that market quotes refer to trades between counterparties with collateral agreement, then we can quite safely assume that the expected value are taken under the risk-free bond numeraire. The pricing formula takes after that one presented above for contracts on other underlying assets, although in this case it is not derived from a replication argument but it is an assertion:

$$\mathbf{FRA}(t;T_{i-1},T_i) = P^D(t,T_i)\tau_i E_D^{T_i}[L_i(T_{i-1}) - K] + \mathbf{LVA}_{\mathbf{FRA}(t;T_{i-1},T_i)} \tag{52}$$

that is the expected Libor rate under the T_i-forward measure of the value of the contract at the expiry T_{i-1}, plus the **LVA**. In (52) $\tau_i = T_i - T_{i-1}$.

The **LVA** in this case is the present value of the difference between the risk-free rate $L_j^D(t)$ and the collateral rate $O_j(t)$, fixed in date t_{j-1}, and valid until date t_j, applied to fraction γ of the value of the contract $\mathbf{FRA}(t_j;T_{i-1},T_2)$ for all the N days between t and the forward settlement T_1, so that $t_N = T_1$:

$$\mathbf{LVA}_{\mathbf{FRA}(t;T_{i-1},T_i)} = \sum_{j=1}^{N} P^D(t,t_j)E_D^{t_j}\Big[\tau_j^C[L_j^D(t) - O_j(t)]\gamma\mathbf{FRA}(t_j;T_{i-1},T_i)\Big] \tag{53}$$

where $\tau_j^C = t_j - t_{j-1}$ is the difference in year fraction between two rebalancing times of the collateral, one day in our case. Formula (52), given the definition of the **LVA** in (53), is recursive. We assume that the market quotes for **FRA**'s refers to the case when **LVA** is nil. This means that the collateral rate is supposed to be the risk-free rate $L^D(t;t_{j-1},t_j) = O(t;t_{j-1},t_j)$, for all j, which is not unreasonable since standard CSA between banks provides for a remuneration of the collateral account at the OIS (or equivalent for other currencies) rate. The OIS rate can be considered also a virtually risk-free rate, or embedding anyway a negligible spread for default risk. If this holds true, then equation (52) reads as:

$$\mathbf{FRA}(t;T_{i-1},T_i) = P^D(t,T_i)\tau_i E_D^{T_i}[L_i(T_{i-1}) - K] \tag{54}$$

so that we retrieve the standard result, as in Mercurio [7], that the **FRA** fair rate is the expected value of the Libor at the settlement date of the contract, under the expiry T_i-forward risk measure:

$$K = L_i(t) = E_D^{T_i}[L_i(T_{i-1})] \tag{55}$$

We have assumed that the market **FRA** settles in T_i, but according to market conventions it actually settles the present value of the pay-off in T_i in T_{i-1}. The market **FRA** fair rate is then different from the "theoretical" rate in (55), since the latter should be corrected by a convexity adjustment as discussed in Mercurio [8]. The adjustment is nevertheless quite small (fraction of a basis point) and can be neglected in typical market conditions, so we will not consider it.

When the collateral agreement provides for a remuneration of the collateral different from the OIS rate, then we have a **LVA** $\neq 0$, and the **FRA** fair rate has to be valued recursively. Let $Q_i(t) = L_i^D(t) - O_i(t)$ be the spread between the daily risk-free rate and collateral rate and assume it is a stochastic process independent from the value of the **FRA**; we can rewrite equation (53) as:

$$\mathbf{LVA}_{\mathbf{FRA}(t;T_{i-1},T_i)} = \sum_{j=1}^{N} P^D(t,t_j) E_D^{t_j}[\tau_j^C Q_j(t)] E_D^{t_j}[\gamma \mathbf{FRA}(t_j;T_{i-1},T_i)] \tag{56}$$

The second expectation in (56) is $P^D(t,T_i)E_D^{T_i}[\gamma(L_i(T_{i-1}) - K)]/P^D(t,t_j)$, so that we finally get:

$$\mathbf{LVA}_{\mathbf{FRA}(t;T_{i-1},T_i)} = \sum_{j=1}^{N} P^D(t,t_j) \left[E_D^{t_j}[\tau_j^C Q_j(t)] \frac{P^D(t,T_i)E_D^{T_i}[\gamma(L_i(T_{i-1}) - K)]}{P^D(t,t_j)} \right] \tag{57}$$

In a very similar fashion, we can derive the the **FVA** for an **FRA**: let $L^F(t;t_{i-1},t_i) = L_i^F(t)$ be the funding rate paid by the bank, with the notation signifying as above. When financing the collateral, *i.e.*: when the NPV of the contract is negative to the bank, it has to pay this rate and receive the collateral, whereas in the opposite situation, *i.e.*: when the NPV is positive, then it invests at the risk-free rate the collateral received, paying the collateral rate. Let $U(t;t_{j-1},t_j) = U_j(t) = L_j^F(t)$- $L_j^D(t)$ be the funding spread over the risk-free rate, and assume it is not correlated with the NPV of the **FRA**. The **FVA** is then:

$$\mathbf{FVA}_{\mathbf{FRA}(t;T_{i-1},T_i)} = \sum_{j=1}^{N} P^D(t,t_j) \left[E_D^{t_j}[\tau_j^C U_j(t)] \frac{P^D(t,T_i)E_D^{T_i}[\gamma(L_i(T_{i-1}) - K)^-]}{P^D(t,t_j)} \right] \tag{58}$$

where $E[X^-] = E[\min(X,0)]$. It is easy to check that:

$$\frac{P^D(t,T_i)E_D^{T_i}[\gamma(L_i(T_{i-1}) - K)^-]}{P^D(t,t_j)} = -\frac{[\gamma\tau_i \mathbf{Floorlet}(t_j;T_{i-1},T_i,K)]}{P^D(t,t_j)}$$

where $\mathbf{Floorlet}(t_j;T_{i-1},T_i,K)$ is the price of a floorlet priced at time t_j, expiry in T_{i-1}, settlement in T_i, and strike K. If the bank has a short position in the **FRA**, then the **FVA** is

$$\frac{P^D(t,T_i)E_D^{T_i}[\gamma(K - L_i(T_{i-1}))^-]}{P^D(t,t_j)} = -\frac{[\gamma\tau_i \mathbf{Caplet}(t_j;T_{i-1},T_i,K)]}{P^D(t,t_j)}$$

where $\mathbf{Caplet}(t_i;T_{i-1},T_i,K)$ is the price of a caplet, and the arguments of the function are the same as for the floorlet.

The total value of the **FRA** is:

$$\mathbf{FRA}(t;T_{i-1},T_i) = P^D(t,T_i)\tau_i E_D^{T_i}[L_i(T_{i-1}) - K] + \mathbf{LVA}_{\mathbf{FRA}(t;T_{i-1},T_i)} + \mathbf{FVA}_{\mathbf{FRA}(t;T_{i-1},T_i)} \tag{59}$$

In any case, the fair rate making zero the value of the contract at inception, has to be computed recursively.

6.2 Interest Rate Swap

Let us now consider an **IRS**: the fixed leg pays a rate denoted by K on dates $T_c^S, ..., T_d^S$ ($\tau_k^S = T_i^S - T_{i-1}^S$). The present value of these payments is obtained by discounting them with the discount curve D. The floating leg receives the Libor fixings on dates $T_a, ..., T_b$, and the present value is also obtained by discounting with the discounting curve D. We assume that the set of floating rate dates include the set of fixed rate dates. The value at time t of the **IRS** is:

$$\mathbf{IRS}(t, K; T_a, ..., T_b, T_c^S, ..., T_c^S) = \left[\sum_{k=a}^{b} P^D(t, T_k)\tau_k L_k(t) - \sum_{j=c}^{d} P^D(t, T_j)\tau_j^S K\right] + \mathbf{LVA}_{\mathbf{IRS}(t;T_a,T_b)} \tag{60}$$

where the **LVA** is defined as:

$$\mathbf{LVA}_{\mathbf{IRS}(t;T_a,T_b)} = \sum_{j=1}^{N} P^D(t, t_j) E_D^{t_j}\left[\tau_j^C [L_j^D(t) - O_j(t)]\gamma \mathbf{IRS}(t_j; T_a, T_b)\right] \tag{61}$$

where $\mathbf{IRS}(t; T_a, T_b) = \mathbf{IRS}(t, K; T_a, ..., T_b, T_c^S, ..., T_c^S)$. The **LVA** is also in this case the difference between the risk-free rate and the collateral rate applied to the fraction γ of the NPV, for all the N days occurring between the valuation date t and the end of the contract $t_N = T_b$.

Also for swaps, we make the assumption that the market quotes refer to the situation when the **LVA** $= 0$, implying that the risk-free and collateral rates are the same. The market swap rate is then the level making nil the value of the contract at the inception T_a:

$$K = S_{a,b}(t) = \frac{\sum_{k=a}^{b} P^D(t, T_k)\tau_k L_k(t)}{\sum_{j=c}^{d} P^D(t, T_j)\tau_j^S K} \tag{62}$$

When risk-free and collateral rates are different, the **LVA** can be evaluated similarly to the case we have examined for the **FRA**. We then have:

$$\mathbf{LVA}_{\mathbf{IRS}(t;T_a,T_b)} = \sum_{j=1}^{N} P^D(t, t_j) E_D^{t_j}[\tau_j^C Q_j(t)] E_D^{t_j}[\gamma \mathbf{IRS}(t_j; T_a, T_b)] \tag{63}$$

The second expectation in (56) is $C_D^{a,b}(t) E_D^{a,b}[\gamma(S_{a,b}(t) - K)]/P^D(t, t_j)$, where $E_D^{a,b}$ is the expectation taken under the swap measure, with numeraire equal to the annuity $C_D^{a,b}(t) = \sum_{j=a+1}^{b} P^D(t, T_j)\tau_j^S$. So we can finally write:

$$\mathbf{LVA}_{\mathbf{IRS}(t;T_a,T_b)} = \sum_{j=1}^{N} P^D(t, t_j)\left[E_D^{t_j}[\tau_j^C Q_j(t)] \frac{C_D^{a,b}(t) E_D^{a,b}[\gamma(S_{a,b}(t) - K)]}{P^D(t, t_j)}\right] \tag{64}$$

The **FVA** can also be defined analogously to the **FRA**'s case, and using the same notation as above, we have:

$$\mathbf{FVA}_{\mathbf{IRS}(t;T_a,T_b)} = \sum_{j=1}^{N} P^D(t, t_j)\left[E_D^{t_j}[\tau_j^C U_j(t)] \frac{C_D^{a,b}(t) E_D^{a,b}[\gamma(S_{a,b}(t) - K)^-]}{P^D(t, t_j)}\right] \tag{65}$$

We can make use of the option on swaps to express the second expectation in (65) as:

$$\frac{C_D^{a,b}(t)E_D^{a,b}[\gamma(S_{a,b}(t)-K)^-]}{P^D(t,t_j)} = -\frac{[\gamma\mathbf{Rec}(t_j;T_a,T_b)]}{P^D(t,t_j)}$$

where $\mathbf{Rec}(t;T_a,T_b)$ is the price of a receiver swaption priced at time t_j, expiry in T_a, on a swap starting in T_a and maturing in T_b, and strike K. If the bank has a short position in the **IRS** (*i.e.*: it is a fixed rate receiver), then the **FVA** is

$$\frac{C_D^{a,b}(t)E_D^{a,b}[\gamma(K-S_{a,b}(t))^-]}{P^D(t,t_j)} = -\frac{[\gamma\mathbf{Pay}(t_j;T_a,T_b)]}{P^D(t,t_j)}$$

where $\mathbf{Pay}(t;T_a,T_b)$ is the price of a payer swaption, and the arguments of the function are the same as for the receiver.

Finally, the total value of the **IRS** is:

$$\begin{aligned}\mathbf{IRS}(t,K;T_a,...,T_b,T_c^S,...,T_c^S) = &\left[\sum_{k=a}^{b} P^D(t,T_k)\tau_k L_k(t) - \sum_{j=c}^{d} P^D(t,T_j)\tau_j^S K\right] \\ &+ \mathbf{LVA}_{\mathbf{IRS}(t;T_a,T_b)} \\ &+ \mathbf{FVA}_{\mathbf{IRS}(t;T_a,T_b)}\end{aligned} \tag{66}$$

At inception, the swap rate $K = S_{a,b}(t)$ is the level making nil the value of the contract and it will be computed recursively from (66).[7]

Example 6.1. *We show an example for an* **IRS***, assuming that the risk-free rate is equal to the Eonia rate; the Euribor forward fixings are at spread over the Eonia. The yearly Eonia foward rates, the spreads and the Euribor forward rates are shown in table 11.*

We price under a CSA agreement with full collateralization ($\gamma = 100\%$) a receiver swap whereby we we pay the Euribor fixing semi-annually (set at the previous payment date) and we receive the fixed rate annually. With market data considered, the fair rate can be easily calculated by means of formula (62) *and it is equal to* 3.3020*. We assume also that we have to pay a funding spread of $U_j(t) = \overline{U} = 15bps$ over the Eonia curve. Finally we assume that the collateral is remunerated at the Eonia rate.*

Under the assumptions above, the **LVA** *of the swap is nil, as it is clear from its definition in* (64)*. The* **FVA** *is different from zero, since there is a funding spread. To compute the* **FVA** *in* (65)*, we have to compute a portfolio of payer swaptions. To this end we make a simplifying assumption that the NPV of the swaptions is constant between two Euribor fixing dates (i.e.: it is constant over periods of six months). The swpations can be computed by means of the volatilities in table 12 with a standard Black formula. It is then possible to plot the profile of the NPV's of the swaptions, which is actually the (approximated) expected negative exposure (***ENE***) of the receiver swap; the profile is plotted in figure 2*

[7] For an analysis of how funding costs should be included in a non collateralized swap, see Castagna [3]. The analysis therein applies here for the non-collateralized fraction of the contract, i.e.: $100\% - \gamma$.

Time	Eonia Fwd	Spread	Fwd Euribor
0	0.75%	0.65%	1.40%
0.5	0.75%	0.64%	1.39%
1	1.75%	0.64%	2.39%
1.5	2.00%	0.63%	2.63%
2	2.25%	0.63%	2.88%
2.5	2.37%	0.62%	2.99%
3	2.50%	0.61%	3.11%
3.5	2.65%	0.61%	3.26%
4	2.75%	0.60%	3.35%
4.5	2.87%	0.60%	3.47%
5	3.00%	0.59%	3.59%
5.5	3.10%	0.59%	3.69%
6	3.20%	0.58%	3.78%
6.5	3.30%	0.58%	3.88%
7	3.40%	0.57%	3.97%
7.5	3.50%	0.57%	4.07%
8	3.60%	0.56%	4.16%
8.5	3.67%	0.56%	4.23%
9	3.75%	0.55%	4.30%
9.5	3.82%	0.55%	4.37%
10	3.90%	0.54%	4.44%

Table 11: Yearly OIS forward rates and spreads over them for forward Euribor fixings.

Swaptions		
Expiry	**Tenor**	**Volatility**
0.5	9.5	27.95%
1	9	28.00%
1.5	8.5	27.69%
2	8	27.09%
2.5	7.5	26.61%
3	7	26.32%
3.5	6.5	26.16%
4	6	26.02%
4.5	5.5	25.90%
5	5	25.79%
5.5	4.5	25.68%
6	4	25.57%
6.5	3.5	25.46%
7	3	25.37%
7.5	2.5	25.28%
8	2	25.22%
8.5	1.5	25.21%
9	1	25.34%
9.5	0.5	25.50%
10	0	

Table 12: Implied volatilities for the portfolio of swpations used to replicated the **ENE** of the receiver swap.

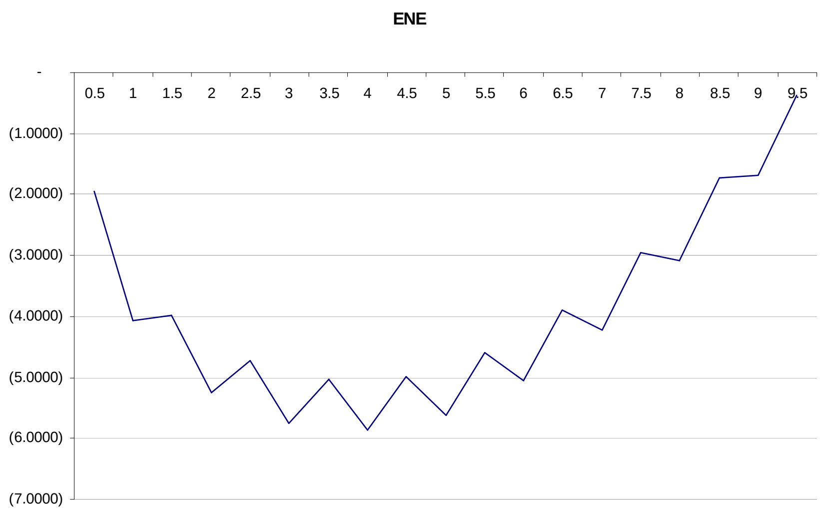

Figure 2: **ENE** of the receiver swap

The results are in table 13. The **FVA** *is quite small for a swap starting at the money, accounting for about half a basis point: an almost negligible impact on the fair swap rate including the funding costs. This rate should be set by a numerical search, and it is the rate making nil the value of the swap, given by the risk-free component plus the* **FVA**, *at inception.*

FVA	-0.0512%
Fair Swap rate	3.3020%
Swap Rate including **FVA**	3.3079%
Difference	0.0059%

Table 13: Fair swap rate, **FVA** and **FVA**-adjusted fair swap rate.

A more conservative **FVA** *can be based on a potential future exposure (***PFE***) rather the expected exposure as we have done above with the* **ENE**. *The* **PFE** *is computed similarly to the* **ENE**, *but considering a level of the future swap rate set at a given confidence level instead of the forward level. We choose* 99% *as for the confidence level.*[8] *The* **PFE** *is plotted in figure 3. Results are shown in table 14. In this case the* **FVA** *is heftier as a percentage of the notional and accounts for about 7 basis points when included in the fair rate.*

The **FVA** *is rather small when the swap starts and it is at-the-money. It can become bigger and bigger as the NPV of the swaps evolves and becomes more negative, or it can*

[8]For a confidence level cl, to determine the corresponding swap rate value at time T we used the equation $S_{a,b}(T) = S_{a,b}(t)\exp[-\frac{\sigma^2}{2}(T-t) + \sigma\sqrt{T-t}\alpha]$, where α is the point of the Normal standard distribution returning a probability cl. In the example, a $cl = 99\%$ implies that $\alpha \approx 2.326$.

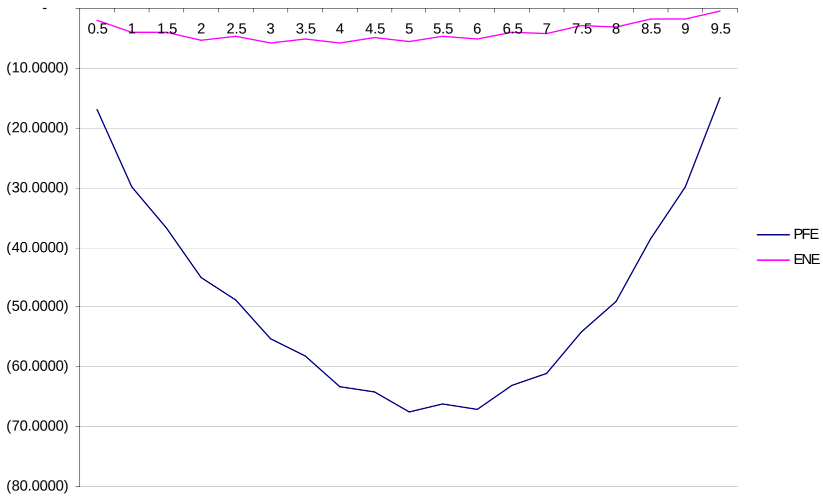

Figure 3: **PFE** of the receiver swap

FVA	- 0.6265%
Fair Swap rate	3.3020%
Swap Rate + Coll. Fund	3.3728%
Difference	0.0708%

Table 14: Fair swap rate, **FVA** and **FVA**-adjusted fair swap rate using **PFE**.

become completely negligible as the NPV increases.

7 Organization of the Dealing Room

In the daily manufacturing of derivative contracts by market-makers, positions are typically hedged so that an offsetting pay-off is synthetically replicated. This happens on an aggregated portfolio level, thus allowing for a natural compensation of exposures originated by the dealing activity.

If the relevant desks operate the replication strategy considering the formula encompassing, for example, the **LVA** (or equivalently, using an effective discount rate accounting for the collateral rate), the final pay-off attained is not equal to the contract's pay-off, as it is manifest from example 2.1. This difference is due to the **LVA** and should be assigned to a Collateral desk, if it exists in the dealing room, to compensate the costs it bears (or the gains it earns) in managing the collateral account. As a consequence the Derivatives desk should try and replicate only the risk-free component of the contract, disregarding the **LVA** and leaving it to the Collateral desk. When trading the contract, the risk-free component of the premium is assigned to the Derivatives desk, while the **LVA** is yielded to the Collateral desk.

By the same token, the **FVA** adjustment should be assigned to the Treasury desk, and to the Repo desk for the repo component if it is present. The **FVA** is the premium that the Derivative desk pays to (or receives from) the other desks involved in the dealing room activity, to be granted an execution of the dynamic replication in a virtually risk-free environment where no collateral and funding effects are operating. In this way, the Derivative desk's performance is gauged on the proper basis, without including contributions others than the correct hedging of the contract's pay-off and the margin that the desk is able to create and to preserve.

On the other hand, the Collateral desk is remunerated (or is charged) with the **LVA** to run its specific activity of management of collateral cash-flows, on which it receives or pays the collateral rate, and it specularly pays or receives the risk-free rate by investing or funding them.

The Treasury desk lends money to and borrows money from the other desks at the risk-free rate. In the money market the Treasury desk pays the funding rate of the bank and it may invest in risk-free assets receiving the risk-free rate. For this activity it is paid the **FVA**.

The Repo desk buys and sells the underlying asset's quantity needed in the dynamic replica. The asset is sold to or bought from the Derivative desk as if it were financed at the risk-free rate. The repo component of the **FVA** is attributed to the Repo desk to account for the difference between the repo rate and the risk-free rate.

Figure 4 shows the decomposition of the total premium in the different components and their attribution to the relevant desks.

In table 15 we show the amount of cash and of underlying asset held by each desk in the replication strategy process. Table 16 shows the same when the underlying asset is bought or sold via repo transactions, so that the Repo desk is involved as well.

As it is quite easy to understand, this has profound implications for the organization of a delaing room. In fact, since recently, desks such as Treasury and Repo, where strongly specialized on linear contracts (deposits, FRAs, repo and reverse repo and so on) and

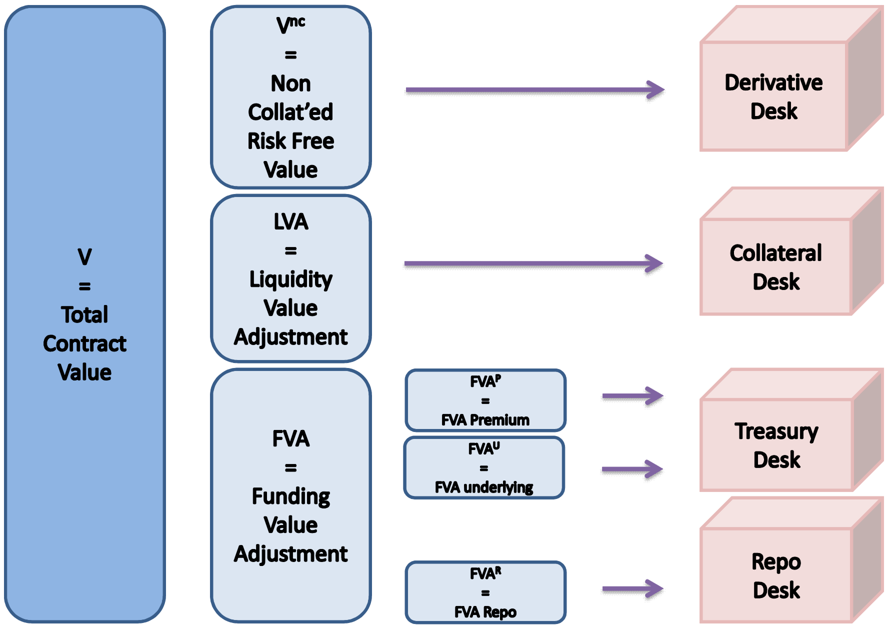

Figure 4: Attribution of the components of a derivative contract value to the relevant desks of a dealing room.

only marginally their skills involved the trading and the risk management of non-linear derivatives contracts, such as options. Nowadays, the importance of funding costs forces these desks to enlarge their skills so as to encompass also the non-linear contracts' risk management, although very likely at a lower level with respect to the specific Derivative desks. The same logic applies also to the Collateral desk, which should be considered not just a cash-flow manager originating from CSA agreements.

This organization can be achieved in two ways, either by training money market and repo traders or by creating Treasury, Repo and Collateral desks with very diffuse competences, gathering traders with a money market and a derivative market making experience. The second option is in our view the easier, quicker and more effective to adopt.

	Und'g Asset	Risk-Free Bond	Bank Bond	Collateral Acc.
Derivative Desk	$\frac{\partial V^{NC}}{\partial S}$	0	0	0
Collateral Desk	$\frac{\partial \mathbf{LVA}}{\partial S}$	0	0	C
Treasury Desk	$\frac{\partial \mathbf{FVA}}{\partial S}$	$(V - C - \Delta S)\mathbf{1}_{\beta>0}$	$(V - C - \Delta S)\mathbf{1}_{\beta<0}$	0
Total Bank	$\frac{\partial V^{C}}{\partial S}$	$(V - C - \Delta S)\mathbf{1}_{\beta>0}$	$(V - C - \Delta S)\mathbf{1}_{\beta<0}$	C

Table 15: Amount of underlying asset, of risk-free bonds and of bank's own bonds held by each desk to dynamically replicate the derivative contract.

	Und'g Asset	Risk-Free Bond	Bank Bond	Collateral Acc.	Repo
Derivative Desk	$\frac{\partial V^{NC}}{\partial S}$	0	0	0	
Collateral Desk	$\frac{\partial \mathbf{LVA}}{\partial S}$	0	0	C	
Treasury Desk	$\frac{\partial \mathbf{FVA^P}}{\partial S}$	$(V-C)1_{\beta>0}$	$(V-C)1_{\beta<0}$	0	
Repo Desk	$\frac{\partial \mathbf{FVA^R}}{\partial S}$			0	$-\Delta S$
Total Bank	$\frac{\partial V^C}{\partial S}$	$(V-C)1_{\beta>0}$	$(V-C)1_{\beta<0}$	C	$-\Delta S$

Table 16: Amount of underlying asset, of risk-free bonds and of bank's own bonds held by each desk to dynamically replicate the derivative contract.

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