# A Framework to Measure and Manage the Prepayment Risk of Mortgages

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# Index

# Possible Approaches to Model Prepayments

- **Empirical Models**:
  - prepayment is modelled as a function of some set of (non-model based) explanatory variables;
  - most of them use either past prepayment rates or some other endogenous variables, such as burnout to explain current prepayment;
  - since they are just heuristic reduced form representations for some true underlying process, it is not clear how they would perform in a different economic environment.
- **Rational Prepayment Models:**
  - they are based on contingent claims pricing theory: prepayment behaviour depends on interest rates' evolution;
  - although these models consistently link valuation and prepayment, their prepayment predictions do not closely match observed prepayment behaviour;
  - in their basic forms these models imply that there will either be no prepayment or all mortgages with similar features will suddenly prepay.

# Facts about Prepayments

Empirical features commonly attributed to mortgage prepayment include:

- some mortgages are prepaid even when their coupon rate is below current mortgage rates;
- some mortgages are not prepaid even when their coupon rate is above current mortgage rates.
- prepayment appears to be dependent on a burnout factor.

The model we propose takes into account these features:

- mortgagors decide whether to prepay their mortgage at random discrete intervals. The probability of a prepayment decision taken on **interest rate reasons**, is commanded by a hazard function $\lambda$: the probability that the decision is made in a time interval of length $dt$ is approximately $\lambda dt$;
- besides refinancing for interest rate reasons, the mortgagors may also prepay for **exogenous reasons** (*e.g.*: job relocation, or sale of the house). The probability of exogenous prepayment is described by a hazard function $\rho$: this represents a baseline prepayment level, the expected prepayment level when no optimal (interest-driven) prepayment should occur.

# The Probability of Prepayment

We model the interest rate based prepayment within a *reduced form* approach. This allows us to include consistently the prepayments into the pricing, the interest rate risk management (ALM) and the liquidity management.

- We adopt a stochastic intensity of prepayment $\lambda$, assumed to follow a CIR dynamics:

$$d\lambda_t = \kappa[\theta - \lambda_t]dt + \nu_t\sqrt{\lambda_t}dZ_t$$

- the intensity is common to all obligors and provides the probability that the mortgage rationally terminates over time;
- the intensity is correlated to the interest rates, so that when rates move to lower levels, more rational prepayments occur;
- the framework is stochastic and it allows for a rich specification of the prepayment behaviour;

The exogenous prepayment is also modelled in a *reduced form* fashion, by a constant intensity $\rho$.

# The Probability of Prepayment

Consider a mortgage with coupon rate $c$ expiring at time $T$:

- each period, given the current interest rates, the optimal prepayment strategy determines whether the mortgage holder should refinance;
- for a given coupon rate $c$, considering also transaction costs, there is a critical interest rates' level $r^*$ such that if rates are lower $(r_t < r^*)$ then the mortgagor will optimally decide to prepay;
- if it is not optimal to refinance, any prepayment is for exogenous reasons;
- if it is optimal to refinance, the mortgagor may prepay either for interest rate related or for exogenous reasons.

These considerations lead to the following prepayment probability:

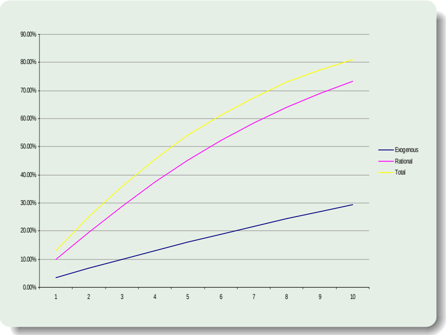
$$\mathrm{PP}(t,T) = 1 - e^{-\rho(T-t)} E\left[e^{-\int_t^T \lambda_s ds}\right] \quad if \quad r_t < r^* \tag{1}$$

$$\mathrm{PP}(t,T) = 1 - e^{-\rho(T-t)} \quad if \quad r_t > r^* \tag{2}$$

# Prepayment Probability Curves: An Example

The figure plots the prepayment probabilities for different times up to the (fixed rate) mortgage's expiry, assumed to be in 10 years. The three curves refer to:

- the exogenous prepayment, given by a constant intensity $\rho = 3.5\%$;
- the rational (interest driven) prepayment, produced assuming $\lambda_0 = 10.0\%$, $\kappa = 27\%$, $\theta = 50.0\%$ $\nu = 10.0\%$.
- the total prepayment *when it is rational to prepay the mortgage* $(r_t < r^*)$.



# Modelling Losses upon Prepayment

Assume at time $t_0$ the bank closes a mortgage deal with a client, with the following contract terms:

- the mortgage notional is $A_0 = 100$;
- the mortgagor pays on predefined scheduled times $t_j$, for $j \in (t_0, t_1, ..., T)$, a fixed rate $c$ computed on the outstanding residual capital at the beginning of the reference period $\tau_j = t_j - t_{j-1}$, denoted by $A_{j-1}$. The interest payment will be then $c\tau_j A_{j-1}$;
- on the same dates, besides interests, the mortgagor pays also $C_j$, which is a portion of the outstanding capital, according to an amortization schedule;
- the expiry in time $T$;
- the mortgagor has the option to end the contract, by prepaying on the payment dates $t_j$ the remaining residual capital $A_j$, together with the interest and capital payments as defined above.

The assumption that the interest, capital, and the prepayment dates are the same, can be easily relaxed.

# Modelling Losses upon Prepayment

- The fair coupon rate can be computed by balancing the present value of future cash-flows with the notional at time $t_0$:

$$\sum_j [c\tau_j A_{j-1} + C_j] P(t_0, t_j) = 100$$

which immediately leads to:

$$c = \frac{100 - \sum_j C_j P(t_0, t_j)}{\sum_j \tau_j A_{j-1} P(t_0, t_j)}$$

where $P(t_0, t_j)$ is the discount factor at time $t_0$ for date $t_j$.

- Assume now that the mortgage is prepaid at a given time $k$; its current value will be:

$$\sum_j [c\tau_j A_{j-1} + C_j] P(t_k, t_j) = A^P$$

where $A^P$ will be almost surely different from the residual capital amount $A_{k-1}$.

# Modelling Losses upon Prepayment

- For the moment we do not consider whether the prepayment has been rational or exogenous. We assume simply that the bank closes a new mortgage similar to the prepaid one, so that this new one replaces all the previous capital payments and yields also new interest rate payments. The fair rate $c^*$ will be determined by the market rates, by the balancing equation as above:

$$\sum_j [c^* \tau_j A_{j-1} + C_j] P(t_k, t_j) = A_{k-1}$$

- The bank will suffer a loss or earn a profit given by:

$$A^P - A_{k-1} = \sum_j P(t_k, t_j) \tau_j A_{j-1} (c - c^*)$$

- The bank is clearly interested to measure (and manage) mainly the expected losses related to the (rational) prepayment at time $t_k$, which we indicate as the *Expected Loss*, or **EL**:

$$\mathbf{EL}(t_k) = E\left[ \max\left[ \sum_j P(t_k, t_j) \tau_j A_{j-1} (c - c^*); 0 \right] \right].$$

# Modelling Losses upon Prepayment

- The **EL** is a function of the term structure of the Libor rates;
- we model Libor rates in a Market Model framework: each Libor forward rate is Lognormally distributed, with a given volatility that can be estimated historically, or extracted from market quotes of caps&floors and swaptions;

$$dF_t = \sigma_t F_t dW_t$$

- the prepayment causing a loss for the bank is mainly originated by the rational prepayment, whose occurrence is commanded by the intensity $\lambda_t$: we assume that this intensity is negatively correlated to the level of the Libor interest rates;
- it is possible to compute the *Expected Loss on Prepayment*, defined as the Expected Loss at a time $t_k$ given that the decision to prepay (for whatever reason) is taken between $t_{k-1}$ and $t_k$:

$$\mathbf{ELoP}(t_k) = E\left[PP(t_{k-1}, t_k)\max\left[\sum_j P(t_k, t_j)\tau_j A_{j-1}(c - c^*); 0\right]\right].$$

# Modelling Losses upon Prepayment

- The framework sketched above is fairly rich so as to take into account all the main aspects of the prepayment effects;
- the computation of the **ELoP** is unfortunately not possible in a closed form formula: this can be a problem since the **ELoP** has to be computed for all the possible exercise dates for a mortgage, and this for a likely large number of mortgages;
- considering the fact that we are interested also in the computation of the sensitivities of the **ELoP**, for hedging purposes, the computation via numerical techniques, such as Montecarlo, can be very time-consuming and unfeasible;
- for this reason we developed an analytical approximation to model:
  - the future fair mortgage rate $c$,
  - the correlation between the fair rate $c^*$ and the rational prepayment intensity $\lambda_t$;
- tests against Montecarlo show that the analytical approximation is satisfying for practical purposes.

# Modelling Losses upon Prepayment

- We are now able to measure which is the expected loss occurring to the bank upon prepayment, at each possible prepayment date $t_k$;
- we define the current value, at time $t_0$, of all the expected losses is the Total Prepayment Cost (**TPC**) related to a mortgage:

$$\mathbf{TPC}(t_0) = \sum_k P(t_0, t_k)\mathbf{ELoP}(t_k)$$

- the **TPC** is the quantity to be hedged. It is a function of:
  - the Libor forward rates,
  - the volatilities of the Libor forward rates,
  - the stochastic rational prepayment intensity $\lambda_t$ and constant exogenous intensity $\rho$;
- the **TPC** can be also included in the mortgage pricing when calculating the fair rate $c$.

# Hedging Prepayment Exposures

The model allows also to compute sensitivities to the main underlying risk factors:

- sensitivity to Libor rates can be computed by tilting each forward a given amount (*e.g.*: 10 bps), and then recomputing the **TPC**;
- Libor exposures can be translated to swap rates exposures, since these are the most liquid and easily tradable hedging tools;
- by the same token, we can compute also the sensitivities to Libor volatilities: these exposures can be hedged by trading caps&floors;
- the sensitivity to the prepayment, both rational and exogenous, can be derived, but no market instrument exists to hedge this exposure. In this case, a VAR-like approach can be adopted and the unexpected cost included into the fair rate, or economic capital posted to cover this risk.

# A Practical Example

Assume 1Y Libor forward rates and their volatilities are those in the table besides. Assume also that the exogenous prepayment intensity is 3% p.a. and the rational prepayment intensity has the same dynamics' parameters as presented above.

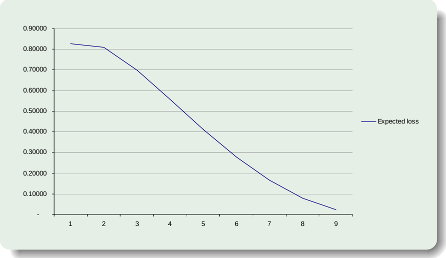
| Yrs | Fwd Libor | Vol |
|---|---|---|
| 1 | 3.50% | 18.03% |
| 2 | 3.75% | 18.28% |
| 3 | 4.00% | 18.53% |
| 4 | 4.25% | 18.78% |
| 5 | 4.50% | 18.43% |
| 6 | 4.75% | 18.08% |
| 7 | 5.00% | 17.73% |
| 8 | 5.25% | 17.38% |
| 9 | 5.50% | 17.03% |
| 10 | 5.75% | 16.78% |

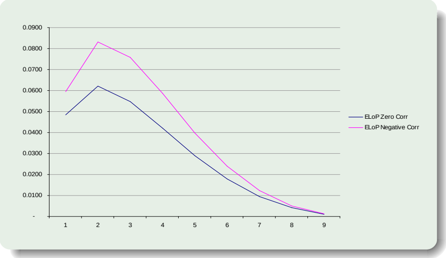
We consider a 10Y mortgage, with a fixed rate paid annually of 3.95%. The fair rate has been computed wihout taking into account any prepayment effect (also credit risk is not considered, although it can be included within the framework). The amortization schedule is in the table besides.

| Yrs | Notional |
|---|---|
| 1 | 100.00 |
| 2 | 90.00 |
| 3 | 80.00 |
| 4 | 70.00 |
| 5 | 60.00 |
| 6 | 50.00 |
| 7 | 40.00 |
| 8 | 30.00 |
| 9 | 20.00 |
| 10 | 10.00 |

# A Practical Example: The **EL** and the **ELoP**.

Given the market and contract data above, we can derive the **EL** at each possible prepayment date, which we assume occurs annually. It is plotted in the figure. The closed form approximation has been employed to compute the **EL**.
In a similar way it is possible to calculate the **ELoP**. We use also in this case an analytical approximation that allows for a correlation between interest rates and the rational prepayment intensity.
In the figure the **ELoP** is plotted for the 0 correlation case and for a negative correlation set at $-0.8$. This value implies that when interest rates decline, the default intensity increases.
Since the loss for the bank is bigger when the rates are low, the **ELoP** in this case is higher than the uncorrelated case.



# A Practical Example: Hedging the Prepayment Risk

The hedging of the prepayment risk (or, of the **TPC**) is possible with respect to the interest rates and to the Libor forward volatilities. The **TPC** is 48 bps.
The first table shows the sensitivity of the **TPC** to a tilt of 10 bps for each forward rate. Those sensitivities are then translated in an equivalent quantity of swaps, with an expiry form 1 year to 10 years, needed to hedge them.

| Yrs | Sensitivity | Hedge Qty |
|---|---|---|
| 1 | 0.02 | 16.82 |
| 2 | 0.02 | 9.07 |
| 3 | 0.02 | 5.58 |
| 4 | 0.01 | 3.45 |
| 5 | 0.01 | 2.19 |
| 6 | 0.01 | 1.38 |
| 7 | 0.01 | 0.92 |
| 8 | 0.00 | 0.64 |
| 9 | 0.00 | 0.40 |

The second table shows the Vega of the **TPC** with respect to the volatilities of each Libor forward rate. Those exposures can hedged with caps&floors, or swaptions in the Libor Market Model setting we are working in (by calibrating the Libor correlation matrix to the swaptions' volatility surface).

| Yrs | Vega |
|---|---|
| 1 | 0.08 |
| 2 | 0.20 |
| 3 | 0.33 |
| 4 | 0.45 |
| 5 | 0.53 |
| 6 | 0.54 |
| 7 | 0.48 |
| 8 | 0.33 |
| 9 | 0.12 |

# Mortgage Pricing

- The model is a powerful tool not only to measure and hedge the prepayment risk, but also to properly take into account the prepayment costs when setting the mortgage fair rate;
- assume we want to include the prepayment cost in the 10 year mortgage we considered before;
- we first include the exogenous prepayment, which is independent form the level of the interest rates, so that on average its effects boil down to an anticipated re-payment of the outstanding notional: this will reduce the fair rate, and accordingly to the data we used above, we have:

$$c = 3.95\% \rightarrow c = 3.89\%$$

- secondly, we include the **TPC** arising from the rational prepayment, which is 48 bps and it surely entails an increase of the fair rate:

$$c = 3.89\% \rightarrow c = 4.00\%$$

  or a total effect of the prepayment of 5 bps in the fair rate;
- should we charge the **EL**, instead of the **ELoP**, to the mortgagor, we would have:

$$c = 3.89\% \rightarrow c = 4.78\%$$

# Mortgage Pricing

- As mentioned above, while the interest rate and volatility risk can be hedged with standard (and liquid) market instruments, the prepayment risk related to the stochasticity of the (rational prepayment) intensity cannot be eliminated;
- one approach is to consider a VaR-like approach:
  - determine the 99% percentile path followed by the intensity, based on the dynamics for $\lambda_t$;
  - recompute the **EL** and the **ELoP** and the **TPC** along this path, by properly considering also the correlation and the volatility of the interest rates;
  - if we apply this approach to the example, the **TPC** is 56 bps and the fair rate is:

    $$c = 3.89\% \rightarrow c = 4.02\%$$

    that means that a generally higher prepayment probability has a low impact on the pricing.

# A Practical Example: Prepayment Probability Curves

In the table we show a comparison between the expected and the 99% percentile rational prepayment.

The higher probability increases the costs, but since it also anticipates the prepayment, the likelihood to have larger differences between current and mortgage rates are reduced.

| Yrs | PP | PP 99% |
|---|---|---|
| 1 | 14% | 19% |
| 2 | 32% | 42% |
| 3 | 49% | 62% |
| 4 | 64% | 77% |
| 5 | 75% | 86% |
| 6 | 83% | 90% |
| 7 | 89% | 92% |
| 8 | 93% | 93% |
| 9 | 95% | 94% |

# A Practical Example: Prepayment Probability Curves

In the table we show three sets of parameters of the rational prepayment intensity dynamics and their effect on:

- the fair rate;
- the fair rate at 99% percentile PP;
- the *tpc*
- the *tpc* at 99% percentile PP.

| | Fair rate | Fair rate 99% |
|---|---|---|
| $\lambda_0 = 50\%, \nu = 0.10\%, \theta = 50\%$ | 4.01% | 4.03% |
| $\lambda_0 = 50\%, \nu = 0.25\%, \theta = 50\%$ | 4.02% | 4.07% |
| $\lambda_0 = 20\%, \nu = 0.25\%, \theta = 20\%$ | 4.01% | 4.04% |

| | **TPC** (bps) | **TPC**99% (bps) |
|---|---|---|
| $\lambda_0 = 50\%, \nu = 0.10\%, \theta = 50\%$ | 50 | 56 |
| $\lambda_0 = 50\%, \nu = 0.25\%, \theta = 50\%$ | 55 | 77 |
| $\lambda_0 = 20\%, \nu = 0.25\%, \theta = 20\%$ | 49 | 65 |

The total effect is rather limited for the mortgage fair rate. When considering the *tpc*, differences are more relevant.

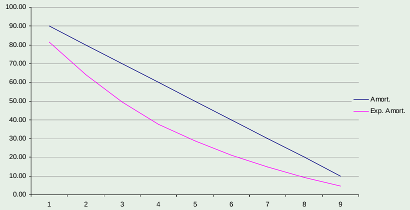
# Parameters Estimation

- The model takes as inputs data that are available in the market:
  - Libor forward rates (typically 6 and 3 month Libor forwards are needed);
  - Libor volatilities, retrieved from caps&floors quotes;
  - Libor correlations, either estimated historically or extracted from a joint calibrations to caps&floors and swaptions quotes.
- The exogenous intensity and the parameters of the rational intensity's dynamics have to estimated from historical data. To this end the following is needed:
  - Database of the mortgage prepayments;
  - Database of the level of the level of the interest rates (to estimate the correlation between the rational prepayment intensity and the level of the interest rates.

# Liquidity Management

We have shown how the framework can be used to measure and hedge the prepayment risk in its form of replacement cost.
The model can be also use to project expected cash flows due to the prepayment activity.
Since the rational prepayment is stochastic and correlated to the level of the interest rate, a VaR-like approach can be adopted also in this case to calculate the maximum and minimum amount of cash flows;

| Expected Cash Flows | Cash Flows | Expected Amort. | Amort. |
|---|---|---|---|
| 22.34 | 13.95 | 81.61 | 90.00 |
| 20.96 | 13.56 | 64.08 | 80.00 |
| 17.59 | 13.16 | 49.44 | 70.00 |
| 14.15 | 12.77 | 37.78 | 60.00 |
| 11.28 | 12.37 | 28.55 | 50.00 |
| 9.13 | 11.98 | 21.08 | 40.00 |
| 7.56 | 11.58 | 14.81 | 30.00 |
| 6.39 | 11.19 | 9.35 | 20.00 |
| 5.48 | 10.79 | 4.46 | 10.00 |



# About Iason

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Iason offers a unique blend of skills and expertise in the understanding of financial markets, in the pricing of complex financial instruments and in the measuring and the management of banking risks. The company's structure is very flexible and grants a fully bespoke service to our Clients.
Iason believes the ability to develop new quantitative finance approaches through research as well as apply those approaches in practice is critical to innovation in risk management and derivatives pricing. It is bringing into the all the areas of the risk management a new and fresh approach based on the balance between rigour and efficiency Iason's people aimed at when working in the dealing rooms.
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