Basel II Second Pillar: an Analytical VaR with Contagion and Sectorial Risks

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Abstract

This paper deals with the effects of concentration (single name and sectoral) and contagion risk on credit portfolios. Results are obtained for the value at risk of the portfolio loss distribution, in the analytical framework originally developed by Vasicek in 1991 [1]. VaR is expressed as a sum of terms: the first contribution represents the value at risk of a hypothetical single-factor homogeneous portfolio, the remaining terms are corrections due to contagion, imperfect granularity and multiple industry-geographic sectors. A detailed numerical analysis is also presented.

1 Introduction

Concentration and contagion risk on credit portfolios have been studied for many years with different methodologies and approaches. Such risks can be seen as departures from the Asymptotic Single-Risk Factor (ASRF) paradigm which underlies the IRB approaches of Basel II [2]. Basic hypothesis of this model include the homogeneity of the underlying portfolio and a common factor driving systematic risk.

In this framework, concentration risk represents a violation of the ASFR model and can be decomposed into two contributions: an idiosyncratic part, *single name* or *imperfect granularity risk*, due to the small size of the portfolio or to the presence of large exposures associated to single obligors and a systematic term, *sectoral concentration*, due to imperfect diversification across sectoral factors. Many portfolio models have been developed in order to deal with concentration risk (e.g. CreditMetrics [3], PortfolioManager [4], CreditPortfolioView [5] etc...) and some of them rely

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on computationally heavy Monte Carlo simulations. A different solution to the problem of calculating economic capital exploits an approximated analytical technique which applies to one-factor Merton type models. This method, originally introduced by Vasicek [1], consists in replacing the original portfolio loss distribution with an asymptotic one, whose value at risk (VaR) can be computed analytically. The difference between the true and the asymptotic VaR can also be computed analytically through a second order approximation [6]. Many steps have been taken in this direction, extending the original Vasicek result for homogeneous portfolios to include granularity risk [7], [8], [9], [10] and sectoral concentration risk (see Pykhtin [11]).

The third source of risk, *credit contagion* lies somewhat in-between the previous two [2]. This risk takes into account the occurrence of default events triggered by inter-dependencies (legal, financial, business-oriented) among obligors. Very diverse approaches have been proposed to tackle this problem. Davis and Lo [12] have built a first model where the default of any company in the portfolio can infect all the others. Egloff *et al* [13] have developed a neural-network inspired model to mimic the structure of links among obligors in a portfolio. Recently Rösch *et al* [14] have proposed an extension of [12] in a default mode scenario where obligors are divided into two categories: those who can be considered immune from contagion "I-firms" (infecting) and those who can be contaminated "C-firms" .

In this paper, we study in a unified framework the effects of concentration and contagion risk. A first attempt to generalize the work by Pykthin in this direction has been pursued by [15]. However, the resulting model specification appears incomplete to some extent and hardly applicable to concrete problems. Here we combine this idea with the contagion specification proposed by Rösch [14], in order to obtain a model which is general enough still preserving analytical tractability.

The paper is organized as follows. We first introduce the model in Section 2, including a detailed specification of the contagion part. We present the core idea of VaR decomposition and the corresponding analytical results in Section 3 and a detailed numerical analysis in Section 4. Section 5 summarizes and collects our final remarks. Technical details and issues can be found in the Appendix.

2 The model

The starting point is a multi-factor default mode Merton model. The portfolio is assumed to have the following features:

- loans are associated to M distinct borrowers. Each borrower has exactly one loan characterized by exposure EAD_i. We define the weight of a loan in the portfolio as $w_i = EAD_i / \sum_{i=1}^{M} EAD_i$
- Each borrower has default probability p_i within a chosen time horizon.
- If borrower i defaults, the amount of loss is determined by its loss-given default, LGD_i. Such quantity is assumed to be stochastic with mean μ_i and standard deviation σ_i and to be independent of the other $LGDs$ and stochastic variables present in the model.

First we introduce the multi-factor set up, without considering contagion, along the lines traced by Pykhtin [11]. Subsequently, we extend the model in order to include the effects of contagion risk.

2.1 Multi-factor model specification

Asset returns $\{X_i\}_{i=1,\ldots,M}$ are assumed to be distributed according to a standard normal distribution and to be linearly dependent on N normally distributed systematic risk factors. Expressing the contribution coming from these sectoral factors in terms of a composite variable $\{Y_i\}_{i=1,\ldots,M}$, one for each borrower, the behavior of the ith borrower's asset return is

$$X_i = r_i Y_i + \sqrt{1 - r_i^2}\xi_i \,. \tag{1}$$

r_i is the sensitivity of borrower i to the systematic risk, $\mathrm{corr}(X_i, X_j) = \rho_{i,j}$ and $\xi_i \sim \mathcal{N}(0,1)$, i.i.d, represents the idiosyncratic component of risk which can be diversified away in the case of an infinitely granular portfolio. The composite factor can be expressed as a linear combination of N independent systematic factors $Z_k \sim \mathcal{N}(0,1)$, $k = 1, \ldots, N$

$$Y_i = \sum_{k=1}^{N} \alpha_{ik}\, Z_k \,, \tag{2}$$

and it is assumed to have unit variance, i.e. $\sum_{k=1}^{N} \alpha_{ik}^2 = 1$. It turns out that asset returns can be cast into the general form

$$X_i = a_i \overline{Y} + \sum_{k=1}^{N} (r_i \alpha_{ik} - a_i b_k) Z_k + \sqrt{1 - r_i^2}\; \xi_i \,. \tag{3}$$

This formula can be analyzed as follows:

- the last term is the idiosyncratic component already discussed.
- The first two terms account for the systematic (sectoral) risk contribution.
  - $\overline{Y} = \sum_{k=1}^{N} b_k Z_k$ is an effective systematic factor, characterized by unit variance i.e. $\sum_{k=1}^{N} b_k^2 = 1$. The coefficients $\{a_i\}$ are effective factor loadings, given by $a_i = r_i \sum_{k=1}^{N} \alpha_{ik} b_k$. They can be derived through an optimization procedure, as shown in [11].
  - The second term $\sum_{k=1}^{N} (r_i \alpha_{ik} - a_i b_k) Z_k$ is independent of $\overline{Y}$ and encodes the conditional asset correlation

$$\rho_{ij}^{Y} = \frac{r_i r_j \sum_{k=1}^{N} \alpha_{ik} \alpha_{jk} - a_i a_j}{\sqrt{(1 - a_i^2)(1 - a_j^2)}} \,. \tag{4}$$

2.2 Credit contagion

Contagion risk can be ascribed to inter-company ties, such as legal (parent-subsidiary) relationships, financial and business oriented relations (supplier-purchaser interactions) and so on. This entails a complex network of links among obligors, which makes the credit contagion problem very hard to solve.

Here we adopt a simplified perspective. We assume that obligors are broadly divided into two categories: those firms which are immune from contagion (referred to as "I-firms", i.e. infecting) and those companies which can be contaminated by the first group through credit contagion ("C-firms"). Asset returns associated to group "I" follow the multi-factor specification given by eq. (1) while "C-firms"' asset returns satisfy

$$X_i = r_i Y_i + \sqrt{1 - r_i^2}\, \xi(\Gamma_i, \epsilon_i)\,. \tag{5}$$

The firm-specific factor $\xi(\Gamma_i, \epsilon_i)$ can be expressed as

$$\xi(\Gamma_i, \epsilon_i) = g_i \Gamma_i + \sqrt{1 - g_i^2}\, \epsilon_i \tag{6}$$

where

- ϵ_i is the usual idiosyncratic contribution,
- the term $g_i \Gamma_i$ encodes the effects of contagion risk. The composite contagion factor Γ_i can be written as a sum over latent contagion variables C_l (assumed to be independent and distributed as $\mathcal{N}(0,1)$)

  $$\Gamma_i = \sum_{l=1}^{N} \gamma_{il}\, C_l\,.$$

  The unit variance property of X_i is preserved if $\sum_{l=1}^{N} \gamma_{il}^2 = 1$. We assume that companies can be mapped into the industry-geographic sector with which they have the highest correlation and we decompose each sector into a "I" segment and a "C" one. Therefore, the contagion effect experienced by an arbitrary "C-firm" can be thought of as the weighted sum of contributions coming from the infecting segments of different sectors. Under this specification, the number of latent contagion factors equals the number of industry-geographic factors, N. The coefficient g_i plays the role of a contagion factor loading and represents a measure of how much obligor i is overall affected by contagion. It is worth noticing that eq.s (5-6) express in compact form also the behavior of "I-firms", with the understanding that $g_i = 0$ in that case. We will come back to the estimation of the contagion parameters in the Appendix.

Having specified the model in this way, by making explicit the effects due to multifactors and contagion, asset returns turn out to follow

$$
\begin{aligned}
X_i &= a_i\overline{Y} + \sum_{k=1}^{N}(r_i\alpha_{ik} - a_i b_k)Z_k + \\
&+ \sqrt{1-r_i^2}\, g_i \sum_{l=1}^{N} \gamma_{il} C_l + \\
&+ \sqrt{1-r_i^2}\,\sqrt{1-g_i^2}\,\epsilon_i\,.
\end{aligned} \tag{7}
$$

The conditional correlation between distinct obligors i and j assumes the form

$$
\rho_{ij}^{Y+C} = \frac{r_i r_j \sum_{k=1}^{N} \alpha_{ik}\alpha_{jk} + \sqrt{1-r_i^2}\,\sqrt{1-r_j^2}\, g_i g_j \sum_{l=1}^{K} \gamma_{il}\gamma_{jl} - a_i a_j}{\sqrt{(1-a_i^2)(1-a_j^2)}}\,. \tag{8}
$$

3 VaR decomposition and results

Given this setup, the portfolio loss rate L can be written as

$$
L = \sum_{i=1}^{M} w_i L_i = \sum_{i=1}^{M} w_i\, \mathbf{1}_{\{X_i \le N^{-1}(p_i)\}}\, LGD_i\,. \tag{9}
$$

Our goal is to calculate the quantile at confidence level q of this quantity, namely $t_q(L)$. It has been shown ([8] and [6]) that such a task can be accomplished through a Taylor expansion around the quantile of another variable $\overline{L}$, such that $t_q(\overline{L})$ is analytically tractable and sufficiently close to $t_q(L)$. The results in the multi-factor case and in the model with contagion are collected in the next two subsections.

3.1 Multi-factor model

Let the variable $\overline{L}$ be defined as the limiting loss distribution in the one-factor Merton framework

$$
\overline{L} = l(\overline{Y}) = \sum_{i=1}^{M} w_i\, \mu_i\, \hat{p}_i(\overline{Y})\,, \tag{10}
$$

where $\hat{p}_i(y)$ is the probability of default of borrower i, conditional on $\overline{Y} = y$

$$
\hat{p}_i(y) = N\left[\frac{N^{-1}(p_i) - a_i y}{\sqrt{1-a_i^2}}\right]\,. \tag{11}
$$

(N indicates the cumulative normal distribution). The quantile of $\overline{L}$ at level q can be calculated analytically as

$$
t_q(\overline{L}) = l(N^{-1}(1-q))\,.
$$

Carrying out the Taylor expansion of $t_q(L)$ around $t_q(\overline{L})$, first-order contributions cancel out and the final result, up to the second order, assumes the form [11]

$$\Delta t_q \equiv t_q(L) - t_q(\overline{L}) = -\frac{1}{2\,l'(y)} \left[\nu'(y) - \nu(y) \left(\frac{l''(y)}{l'(y)} + y \right) \right] \Bigg|_{y=N^{-1}(1-q)} \tag{12}$$

The function $l(y)$ is defined as in (10), while $\nu(y) = \mathrm{var}[L|\overline{Y} = y]$ is the conditional variance of L on $\overline{Y} = y$. This function can be further decomposed in terms of its systematic and idiosyncratic components

$$\nu(y) = \nu_\infty(y) + \nu_{GA}(y)\,, \tag{13}$$

where

$$\begin{aligned} \nu_\infty(y) &= \mathrm{var}[E(L|\{Z_k\})|\overline{Y} = y] = \\ &= \sum_{i=1}^{M} \sum_{j=1}^{M} w_i w_j \mu_i \mu_j \left[N_2(N^{-1}[\hat{p}_i(y)], N^{-1}[\hat{p}_j(y)], \rho_{ij}^Y) - \hat{p}_i(y)\hat{p}_j(y) \right] \,, \end{aligned} \tag{14}$$

$$\begin{aligned} \nu_{GA}(y) &= E[\mathrm{var}(L|\{Z_k\})|\overline{Y} = y] = \\ &= \sum_{i=1}^{M} w_i^2 \left(\mu_i^2 \left[\hat{p}_i(y) - N_2(N^{-1}[\hat{p}_i(y)], N^{-1}[\hat{p}_i(y)], \rho_{ii}^Y) \right] + \sigma_i^2 \hat{p}_i(y) \right) \,. \end{aligned} \tag{15}$$

($N_2(.,.,.)$ is the bivariate normal cumulative distribution function). The first term, $\nu_\infty(y)$, accounts for the correction to the loss distribution due to the multi-factor setting, in the limit of an infinitely fine-grained portfolio. $\nu_{GA}(y)$ is the granularity adjustment term. The quantity ρ_{ii}^Y is obtained by replacing the index j with i in eq. (4). In the special case of homogeneous LGDs and default probabilities p_i, it becomes proportional to the Herfindahl-Hirschman index $HHI = \sum_{i=1}^{M} w_i^2$ (see [9]).

Explicit expressions for the derivatives of $l(y)$ and $\nu(y)$ can be found in the Appendix.

3.2 Contagion effects

Repeating the analysis performed in the previous paragraph, the results obtained for the multi-factor set up can be easily extended in order to include contagion risk.

Eq. (12) is still valid, with the understanding that now the conditional variance $\nu(y)$ must be replaced by

$$\nu^C(y) = \nu_\infty^C(y) + \nu_{GA}^C(y)\,,$$

where

$$\begin{aligned} \nu_\infty^C(y) &= \mathrm{var}[E(L|\{Z_k, C_l\})|\overline{Y} = y] = \\ &= \sum_{i=1}^{M} \sum_{j=1}^{M} w_i w_j \mu_i \mu_j \left[N_2(N^{-1}[\hat{p}_i(y)], N^{-1}[\hat{p}_j(y)], \rho_{ij}^{Y+C}) - \hat{p}_i(y)\hat{p}_j(y) \right] \,, \end{aligned} \tag{16}$$

$$\begin{aligned}\nu^C_{GA}(y) &= E[\mathrm{var}(L|\{Z_k, C_l\})|\overline{Y}=y] = \qquad (17)\\ &= \sum_{i=1}^{M} w_i^2 \left(\mu_i^2 \left[\hat{p}_i(y) - N_2(N^{-1}[\hat{p}_i(y)], N^{-1}[\hat{p}_i(y)], \rho_{ii}^{Y+C})\right] + \sigma_i^2\, \hat{p}_i(y)\right) .\end{aligned}$$

In this case, ν^C_∞ encodes the correction to the loss distribution due to multi-factor *and* contagion effects, in the limit of an infinitely granular portfolio. ν^C_{GA} represent the granularity contribution. The conditional correlation appearing in eq.s (16-17) is now given by formula (8).

4 Applications

This section is devoted to applications of the theoretical model and numerical examples. In the first subsection we show and comment upon the results we get, highlighting the role played by the numerous parameters which specify the model. In the second part, we compare the values of the value at risk we obtain with those derived through Monte Carlo simulations by Gordy (2000) [16] and Carey (2001) [17]. As it will be evident, the agreement is very satisfactory.

4.1 Numerical analysis

We aim at showing how the value at risk calculated through the approximated formula at the second order, eq. (12), behaves as a function of the parameters which define the model. In particular, we focus on the number of obligors M, the number of systematic risk factors N (which also affects the contagion specification) and the rating quality of the portfolio. However, before entering the details, some general information about the characterization of the portfolio and the model itself is needed.

Portfolio data and features of the model

- Loan exposures are assigned following the rule $EAD_i = i^3$ [9]. For $M \sim 100$ and above, the loan to one borrower limit of 4% of the total portfolio size is not exceeded.
- We sort obligors in ascending order with respect to their exposure. We further assume that the last 20% of them belongs to the group of infecting "I-firms".
- In the following, we assume for sake of simplicity that industry-geographical sectors are already expressed in terms of independent standardized normal risk factors (the $\{Z_k\}$ variables of the theoretical model)[1].

  The information about the dependence of single obligors on these factors is encoded into obligor/sector correlation coefficients, i.e. $\rho_{ik} = r_i \alpha_{ik}$, where i

[1]The most general case of dependent sectors can be reduced to this simplified scenario by means of a Cholesky decomposition of the variance-covariance matrix or through the principal component analysis (PCA).

represents the i-th obligor and k the sector. Assuming that banks have these kind of data at their disposal, we further simplify the model by introducing "classes of correlation". For instance, if we only considered three classes (low, average and high correlation), we would assign all correlations below 33% the value $\rho_{ik} = 16.5\%$, those between 34% and 66%, $\rho_{ik} = 49.5\%$ and so on. Given the coefficients ρ_{ik}, the factor loadings r_i can be easily deduced by observing that $r_i^2 = \sum_{k=1}^{N} \rho_{ik}^2$. In an analogous way, α_{ik} can be derived from $\rho_{ik} = r_i \alpha_{ik}$. (In the following analysis we consider five "correlation classes".)

- Very similar observations can be applied to contagion factors. For simplicity, also latent variables C_l are assumed to be $\sim \mathcal{N}(0,1)$ i.i.d.

  The coefficients γ_{ik}, which encode the dependence of single obligors on the infecting segments of each industry-geographic area, are grouped into classes as well. However, they satisfy a different constraint, $\sum_{k=1}^{N} \gamma_{ik}^2 = 1$. (Also in this case, we opt for five "contagion classes".)

  In the contagion specification, we are left with another parameter, to be chosen at each bank's discretion: the factor loading g_i. Similarly, we group its possible values into five classes.

- As far as the quality of the portfolio is concerned, we consider three different possibilities, based on a seven rating classes subdivision (see Gordy 2000 [16] for details):

  1. *high quality* portfolio, characterized by a percentage of speculative grade loans (BB and below) less than 25%,
  2. *average quality* portfolio, such that speculative grade loans account for 50% of the total exposure,
  3. *low quality* portfolio, made up of roughly 79% of speculative grade loans.

  This can be visualized in Fig. 1:

- Eventually, following the specification introduced by Gordy 2000 [16], we choose a constant LGD mean value, across different rating classes, given by $\mu_i = 30\%$. The corresponding standard deviation turns out to be $\sigma_i = 1/2\sqrt{\mu_i(1-\mu_i)} \sim 0.229$.

We now address the core discussion of the numerical analysis.

VaR *vs* number of obligors

We assume an *average quality* portfolio, characterized by $N = 5$ industry-geographic sectors, and, correspondingly, five latent contagion risk factors. We fix the confidence level at $q = 99.5\%$ and calculate the value at risk for M, the number of obligors, ranging from 100 to 1000. We store the results in Table 1:

The first column collects the Hirschmann-Herfindahl index (HHI) for each scenario, giving an idea of the granularity of the underlying portfolios. As it appears clearly, upon increasing the number of loans, M, such a contribution becomes less relevant.

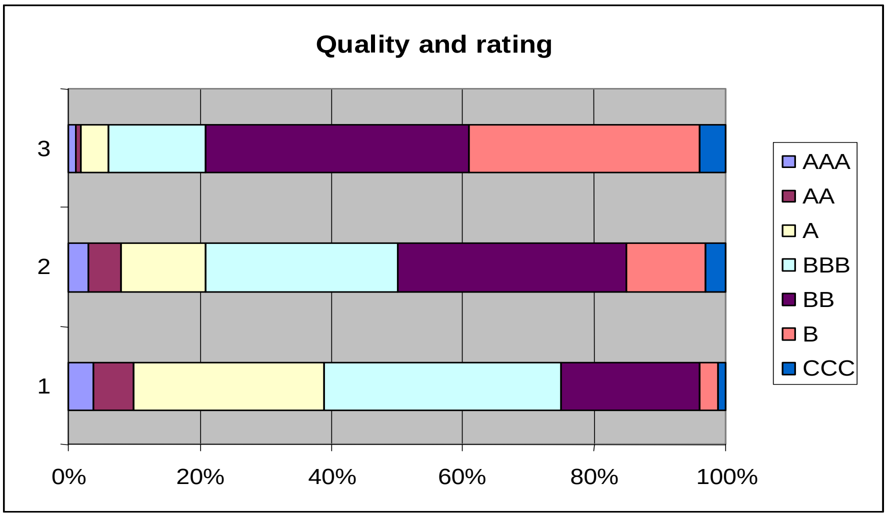

Figure 1: Subdivision into seven rating classes for a high (1), average (2) and poor (3) quality portfolio.

	HHI	$VaR_{99.5\%}$	Δ_{YC}	Δ_C	Δ_{GA}	Δ_∞	cpu-time
M=100	0.0227	0.0386	0.0175	0.0014	0.0151	0.0024	41.0190
M=200	0.0114	0.0288	0.0112	0.0022	0.0079	0.0034	162.8141
M=500	0.0046	0.0252	0.0059	0.0020	0.0030	0.0029	1020.0000
M=1000	0.0023	0.0225	0.0048	0.0024	0.0015	0.0033	4100.0000

Table 1: Results obtained for an average quality portfolio, characterized by seven rating classes, five industry-geographic areas and contagion factors, at the level of confidence $q = 99.5\%$.

The following columns contain respectively the value at risk of the loss distribution L, given by the approximated formula (12) in the presence of sector and contagion risk, and the correction to the homogeneous single-factor asymptotic VaR, decomposed into its main contributions:

- Δ_{YC} = total correction due to multi-factor and contagion;
- Δ_C = total correction due to contagion;
- Δ_{GA} = granularity adjustment;
- Δ_∞ = correction due to multi-sector and contagion, for a homogeneous portfolio.

Results from columns (2) and (6) are represented graphically in Fig. 2

As it emerges from the picture, the value of $VaR_{99.5\%} \equiv t_{99.5\%}(L)$ decreases of about 160 basis points, moving from 100 obligors to 1000. This behavior is consistent with the fact that the portfolio becomes more homogeneous as the number of loans increases, leading to a diminishing granularity adjustment.

The corrections due to the multi-sector set up and the effects of contagion are only mildly affected by the number of obligors.

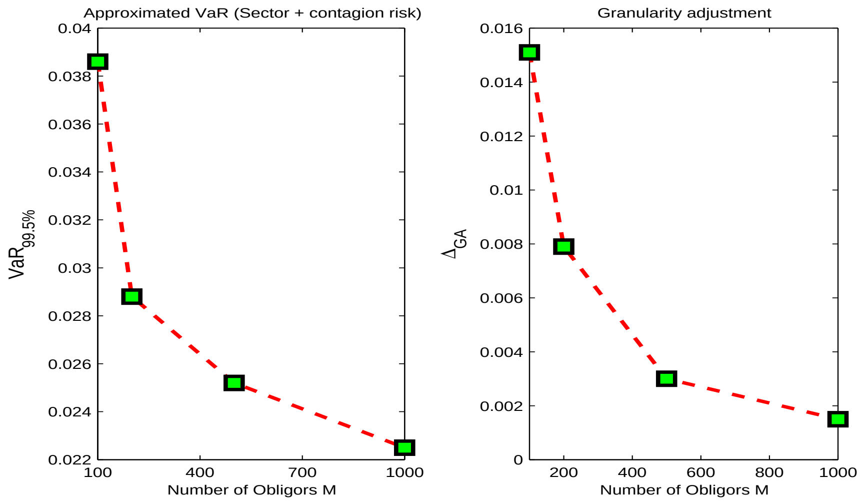

Figure 2: Approximated $VaR_{99.5\%}$ and granularity adjustment as a function of M (number of obligors), for an average quality portfolio, characterized by seven rating classes, five industry-geographic sectors and contagion factors.

VaR *vs* number of sectors

We now let the number of systematic risk factors vary from 1 to 10. The results for an average quality portfolio, characterized by $M = 300$ obligors ($HHI = 0.0076$) and level of confidence $q = 99.5\%$ are summarized in Table 2:

	$t_q(\overline{L})$	$VaR_{99.5\%}$	Δ_{YC}	Δ_C	Δ_{GA}	Δ_∞
N=1	0.0370	0.0442	0.0072	0.0035	0.0036	0.0037
N=2	0.0300	0.0371	0.0071	0.0029	0.0038	0.0033
N=6	0.0153	0.0238	0.0085	0.0016	0.0060	0.0025
N=10	0.0116	0.0226	0.0110	0.0018	0.0081	0.0029

Table 2: Results for and average quality portfolio, characterized by seven rating classes, $M = 300$ obligors ($HHI = 0.0076$) and level of confidence $q = 99.5\%$.

The first column displays the data corresponding to the homogeneous, single-factor VaR, $t_q(\overline{L})$. As it appears clearly, upon increasing the number of sectors, diversification benefits turns out to play a central role. This effect is particularly evident in the asymptotic component $t_q(\overline{L})$, thanks to the effective factor loading a_i, which takes into account the combined contribution of all sectors (see Section 2). The result is shown in Fig. 3.

We eventually notice that the correction Δ_∞, directly related to the multi-sector set up, assumes almost constant values. This is consistent with our choice, at the level of simulation, of picking N sectors *globally*, but requiring each obligor to interact with at most two industry-geographic areas simultaneously. Similarly, we have chosen each "C-firm" to be infected by at most three infecting segments beyond the one in its own

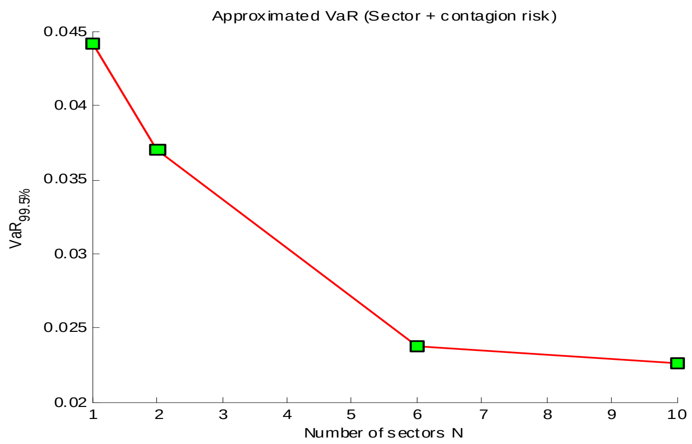

Figure 3: Approximated $VaR_{99.5\%}$ as a function of N (number of sectors) for an average quality portfolio, characterized by seven rating classes and $M = 300$ obligors ($HHI = 0.0076$).

sector. Therefore, also the effects of contagion risk my appear somehow "diluted", as the number of factors N increases.

VaR as a function of the portfolio quality

We conclude by analyzing the effects of rating on the value at risk. We consider $M = 300$ obligors, $N = 6$ systematic and contagion risk factors, and different values of q. We refer to Fig. 1 for the quality properties of the portfolios, in terms of seven rating classes.

We collect the results in Tables 3-5.

High Qlty	$VaR_{99.5\%}$	Δ_{YC}	Δ_C	Δ_{GA}	Δ_∞
q=99%	0.0099	0.005	0.00093	0.0037	0.0013
q=99.5%	0.0114	0.0055	0.00098	0.0041	0.0015
q=99.9%	0.0156	0.0068	0.001	0.005	0.0018

Table 3: Results for a high quality portfolio (25% of BB and below), characterized by seven rating classes, $M = 300$ obligors ($HHI = 0.0076$).

Ave Qlty	$VaR_{99.5\%}$	Δ_{YC}	Δ_C	Δ_{GA}	Δ_∞
q=99%	0.021	0.0078	0.0015	0.0055	0.0023
q=99.5%	0.0238	0.0085	0.0016	0.006	0.0025
q=99.9%	0.0307	0.0103	0.0017	0.0075	0.0029

Table 4: Results for an average quality portfolio (50% of BB and below), characterized by seven rating classes, $M = 300$ obligors ($HHI = 0.0076$).

Low Qlty	$VaR_{99.5\%}$	Δ_{YC}	Δ_C	Δ_{GA}	Δ_∞
q=99%	0.0324	0.0098	0.0027	0.0061	0.0037
q=99.5%	0.0363	0.0108	0.0029	0.0068	0.004
q=99.9%	0.0453	0.0131	0.0031	0.0088	0.0044

Table 5: Results for a low quality portfolio (79% of BB and below), characterized by seven rating classes, $M = 300$ obligors ($HHI = 0.0076$).

The results are consistent with the fact that, upon increasing the quality of portfolio loans, VaR_q decreases as well as the correction with respect to the asymptotic value at risk.

4.2 Comparative analysis with Monte Carlo simulations

We conclude the numerical analysis by comparing our results with those obtained through Monte Carlo simulations. In particular, we refer to the works by Carey (2001) [17] and Gordy (2000) [16]. The former offers a very detailed description of the simulation and the underlying portfolio, the latter presents a comparative analysis between the the CreditMetrics [3] and CreditRisk+ [18] models.

Before entering the details of the discussion, we must stress that the works we refer to use an approach which is not based on the decomposition of the value at risk into its asymptotic part and corrections (they actually appeared before the huge literature on this topic). Their methodological framework relies on the simulation of the “true” loss distribution, through generation of numerous scenarios. However, our model, by construction, is flexible enough to allow for a satisfactory correspondence with both [17] and [16].

Carey 2001

Carey assumes a portfolio with the following features (see Table 1 in [17]):

- *default-mode* credit model,
- flexible number of obligors, but close to 500,
- maximum loan to one borrower limit in the exposure of about 3%,
- portfolio quality as expressed in Table 6 (following Moody’s criteria),

	Rating
>A	20%
Baa	30%
Ba	35%
B	15%

Table 6: Portfolio decomposition into rating classes used by [17].

- *LGD* characterized by mean value $\mu = 37\%$, constant across rating classes.

Setting up a comparable (though not perfectly matched) portfolio, fixing $M = 500$ and letting the confidence level q vary, we obtain the following results for the value at risk:

	$q = 95\%$	$q = 99\%$	$q = 99.5\%$	$q = 99.9\%$
Carey	0.0187	0.0271	0.0304	0.0387
Modello	0.0153	0.0229	0.0263	0.0343

Table 7: Comparison between the analytical results of our model and the outcomes of the Monte Carlo simulation, performed by Carey [17]

Despite the impossibility to reproduce exactly the portfolio used by Carey, the two models produce values of the VaR which agree on the second digit, the discrepancy being of about $30 \div 40$ basis points.

Gordy 2000

We now compare our model with those analyzed by Gordy [16], namely

- the so called "restricted" CreditMetrics (CM2S), which only accounts for default events, without considering migrations across rating classes,
- a version of CreditRisk+ (CR+) [18], characterized by a single systematic risk factor, distributed according to a gamma distribution, with unitary mean and standard deviation σ. We stress that the parameter σ can be chosen in an arbitrary way, affecting the final value of VaR to a certain degree.

Gordy considers portfolios of $M = 5000$ loans, of different credit quality, according to the distributions into rating classes proposed in Fig. 1. The mean value of the loss given default is taken equal to $\mu_i = 30\%$. Systematic factor loadings r_i are set to particular values, collected in Table 2, [16]. The numerical analysis is then performed for two different scenarios: the homogeneous portfolio case and in the presence of imperfect granularity.

We can reproduce quite accurately the data of the homogeneous case. We assume a portfolio which match closely that by Gordy, except for the number of obligors (we fix $M = 400$). The results are summarized in Table 8.

	Model	CM2S	CR+ $\sigma = 1.5$
$q = 99.97\%$	0.0264	0.02649	0.03187
$q = 99.5\%$	0.0174	0.01747	0.02009
$q = 99\%$	0.0153	0.01527	0.01728

Table 8: Comparison between the VaR of our model and those obtained by Gordy [16], for a homogeneous portfolio.

The agreement, especially with the CM2S model, obtained through Monte Carlo simulation, is excellent.

We conclude, by comparing the results for a portfolio with granularity. Gordy proposes a calibration of the exposures, based on rating [16]. We limit ourselves to keep our specification of granularity, previously exposed (for $M = 400$ obligors), and collect the results in Tables 9-11:

	Model	CM2S	CR+ $\sigma = 1$	CR+ $\sigma = 1.5$	CR+ $\sigma = 4$
$q = 99.97\%$	0.0341	0.02714	0.02736	0.03225	0.05149
$q = 99.5\%$	0.0226	0.01795	0.01847	0.02033	0.02488
$q = 99\%$	0.0199	0.01578	0.01628	0.01749	0.01916

Table 9: Comparison between the VaR of our model and those obtained by Gordy [16], for an average quality portfolio with granularity.

	Model	CM2S	CR+ $\sigma = 1$	CR+ $\sigma = 1.5$	CR+ $\sigma = 4$
$q = 99.97\%$	0.0462	0.04558	0.04877	0.05770	0.09251
$q = 99.5\%$	0.0323	0.03124	0.03320	0.03664	0.04504
$q = 99\%$	0.0288	0.02782	0.02936	0.03161	0.03481

Table 10: Comparison between the VaR of our model and those obtained by Gordy [16], for a poor quality portfolio with granularity.

	Model	CM2S	CR+ $\sigma = 1$	CR+ $\sigma = 1.5$	CR+ $\sigma = 4$
$q = 99.97\%$	0.0198	0.01342	0.01277	0.01490	0.02345
$q = 99.5\%$	0.0125	0.00847	0.00850	0.00928	0.0121
$q = 99\%$	0.0108	0.00733	0.00745	0.00794	0.00858

Table 11: Comparison between the VaR of our model and those obtained by Gordy [16], for a high quality portfolio with granularity.

Despite the fact that our model and those analyzed by Gordy are specified in a different way, we obtain values of the VaR which are compatible, reaching a satisfactory agreement in the case of a low quality portfolio.

5 Conclusions

In this paper we have shown how to compute analytically the value at risk for a portfolio of loans, non homogeneous in the exposures and in the presence of *both* multiple industry-geographic sectors *and* contagion risk.

The key idea consists in approximating the "true" VaR as a sum of terms: the first contribution is the asymptotic VaR, pertaining to the limiting case of a single-factor homogeneous portfolio (ASRF), the remaining terms are the corrections due to granularity and the multi-factor set up. Contagion risk affects the adjustments, but do not have any impact on the asymptotic component of value at risk.

An important aspect of the model proposed is that it allows to obtain good estimates of the value at risk, without relying on time consuming Monte Carlo simulations.

Appendix

A Contagion parameters

The parameters to be estimated from market data are the factor loadings $\{g_i\}$ and the coefficients $\{\gamma_{il}\}$ which appear in the expansion of the composite contagion factor Γ_i in terms of the latent variables C_l. The idea we propose in our model specification, in order to choose such parameters, is to rely on the information encoded into the revenues generated by single obligors.

We assume that data about the revenues of each obligor, R_i, are available. In particular, we assume it is possible to quantify the amount of revenues earned from transactions with the infecting segment of each sector. Let us call this quantity R^I_{ik}, where $k = 1, \ldots, N$ specifies the sector and $i = 1, \ldots, M$ the single obligor.

The coefficient γ_{ik} can be expressed in terms of the revenues data as follows

$$\gamma_{ik} = \mathcal{C}\,\frac{R^I_{ik}}{R_i}\,,$$

where the proportionality constant is set to the value

$$\mathcal{C} = \frac{1}{\sqrt{\sum_{k=1}^{N}\left(\frac{R^I_{ik}}{R_i}\right)^2}}\,.$$

In this framework, we assume the factor loading g_i to be a discretionary parameter, which measures the overall sensitivity of obligor i to contagion risk.

B Derivatives of $l(y)$ and $\nu(y)$, eq. (12)

In order to calculate Δt_q according to eq. (12), we also need explicit expressions for the derivatives of $l(y)$ and $\nu(y)$. Given eq.s (10), (11), (14) and (15), the derivatives read

$$l'(y) = \sum_{i=1}^{M} w_i\,\mu_i\,\hat{p}'_i(y)\,,$$

$$l''(y) = \sum_{i=1}^{M} w_i\,\mu_i\,\hat{p}''_i(y)\,,$$

$$\hat{p}'_i(y) = -\frac{a_i}{\sqrt{1-a_i^2}}\,n\left[\frac{N^{-1}(p_i) - a_i y}{\sqrt{1-a_i^2}}\right]\,,$$

$$\hat{p}_i''(y) = -\frac{a_i^2}{1-a_i^2}\,\frac{N^{-1}(p_i)-a_i y}{\sqrt{1-a_i^2}}\; n\left[\frac{N^{-1}(p_i)-a_i y}{\sqrt{1-a_i^2}}\right],$$

$$\nu_\infty'(y) = 2\sum_{i=1}^{M}\sum_{j=1}^{M} w_i w_j \mu_i \mu_j\, \hat{p}_i'(y)\left[N\left(\frac{N^{-1}[\hat{p}_j(y)] - \rho_{ij}^{Y} N^{-1}[\hat{p}_i(y)]}{\sqrt{1-(\rho_{ij}^{Y})^2}}\right) - \hat{p}_j(y)\right],$$

$$\nu_{GA}'(y) = \sum_{i=1}^{M} w_i^2 \hat{p}_i'(y)\left(\mu_i^2\left[1-2N\left(\frac{N^{-1}[\hat{p}_i(y)] - \rho_{ii}^{Y} N^{-1}[\hat{p}_i(y)]}{\sqrt{1-(\rho_{ii}^{Y})^2}}\right)\right] + \sigma_i^2\right).$$

In the presence of credit contagion, all of the above formulas are easily generalized by replacing ρ_{ij}^{Y} with ρ_{ij}^{Y+C}.

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