Funding Valuation Adjustment (FVA) and Theory of the Firm: A Theoretical Justification of the Inclusion of Funding Costs in the Evaluation of Financial Contracts*

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Abstract

We propose a justification of the current practice to charge funding costs in the price of the investments of a bank (and more generally of a firm) within the classical theory of the firm originated from the works by Modigliani and Miller [8] and Merton [5]. This is a first attempt to generalise these works leading to interesting, but not completely sound, results.

1 Introduction

In the financial industry, in the last year a lively debate started over the theoretical justification of the widely adopted market practice to include funding costs in the valuation of assets and, more generally, of derivatives contracts (the so called Funding Valuation Adjustment, **FVA**).

Hull and White, in an article appeared on Risk magazine [3], argued that the **FVA** is not justified and it is not consistent with very well know theoretical results that awarded a Nobel prize to their authors, namely Modigliani and Miller [8] and Merton [5]. Since practitioners never really followed the indication provided by these theoretical results, some responses to the Hull and White's article appeared, including our own in [2].

After that, the two authors tried to better explain and sustain their claim in a more theoretically detailed paper [4]: we think the arguments presented in this second work are still not conclusive. In any case, in our opinion, all the **FVA** debate is still missing

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some points that can make the standard framework by Modigliani, Miller and Merton consistent with the current market practice. In this note we will show how the inclusion of funding costs, and more generally of risk adjusted pricing based on measure such as the RAROC, are fully justified when the assumptions of the standard theory are relaxed to take into account typical situations occurring in reality.

In more details, we will show that when the Modigliani-Miller-Merton framework is extended to a multi-period setting, in which shareholders are risk averse towards the default of the firm, the inclusion of funding costs and, in general, of the **FVA**, is perfectly supported by the theoretical analysis.

2 Value of the Firm and Returns on the Balance Sheet Items

Assume we are in a single period setting, starting in t and ending in T. Let $A(t)$ be the price of a security at time t held in the assets of the balance sheet: it is a stochastic variable distributed at future time T as $A(T) \sim f(\mu, \sigma)$, where f is a distribution function with mean μ and variance σ.

Furthermore, let K be the face face value of the debt expiring in T: the risk-free discounted value at time t is $D^*(t) = Ke^{-r(T-t)}$, where r is the risk-free rate. The market price of the debt is $D(t) = D^*(t) - p$, where p is the compensation for the default risk. The value of the equity in t is $E(t)$. By balance sheet equivalence:

$$A(t) = D(t) + E(t)$$

states that the value of the assets equals the value of the liabilities.

The classical article by Modigliani and Miller (MM) [8] shows that, in a perfect and frictionless market, the composition of the capital structure (*i.e.*: the funding mix of equity and debt) does not affect the value of the firm, which is ultimately only given by the return of the asset A, deducted the cost of capital. MM argue that in efficient markets the total weighted cost of the capital is constant and independent of the financial leverage D/E, so that the firm's net revenues are independent from the financial leverage as well.

Merton [5] shows that even when including the possibility of the firm's default in the framework, the MM's result holds. In a single period setting ending in T, given that the value of the equity is a function of the value of the firm with terminal condition $E(T) = \max[A(T) - D^*(T), 0] = \max[A(T) - K, 0]$, Merton shows that one can build at time t a portfolio perfectly hedged earning the risk free rate r by trading suitable quantities of the firm's asset $A(t)$, equity $E(t)$ and risk-less bonds $D^*(t)$. From the hedged portfolio, one can derive the value of the equity at time t in a risk-neutral setting, since no real world parameter appears in the evaluation equation.

It is manifest that the value of the equity is equivalent to the value of a call option on the asset A with strike K. Assuming we are working in a Black&Scholes (B&S) economy, the risk neutral stochastic process followed by the assets A is:

$$dA_t = rA_t dt + \sigma A_t dW_t \tag{1}$$

The equity $E(t)$ is derived by means of the B&S call option formula, under the (martingale equivalent) risk neutral measure:

$$E(t) = A_t N(d_1) - Ke^{-r(T-T)} N(d_2) \tag{2}$$

where $N()$ is the Normal distribution function and

$$d_1 = \frac{\ln \frac{A_t}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}; \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

The value of the firm's debt can be easily derived from the balance sheet equivalence of assets and liabilities, so that $D(t) = A(t) - E(t)$. Since $D(t) = D^*(t) - p = Ke^{-r(T-T)} - p$, the premium for the default risk embedded in the value of the firm's debt, p, is obtained through the put-call parity: $p = A(t) - Ke^{-r(T-T)} + E(t)$.

Once the value of the equity and of the debt are derived, Merton shows that the assets of two firms, one with zero financial leverage and the other with an amount of debt D, must have the same value or otherwise an arbitrage can be set up.

In the Merton's framework the risk-neutral expected return of the equity is the risk-free rate r, by construction of the perfectly hedged portfolio including the asset, the equity and the default risk free debt; also the expected return of the firm's debt, after accounting for the expected losses upon default, is the risk-free rate r. This means that the MM theorem is trivially proved under the risk-neutral measure, since the weighted average cost of the capital (WACC) is always r, independently from the capital structure. This result stems from the replication argument resembling the one used to evaluate standard option, which earns the risk-free rate.

One can argue that actually it is quite difficult to trade directly in the asset of the firm, so that the set up of a perfectly hedged portfolio is not possible. Considering only the real world measure, the Merton's framework is not invalidated: the expected return of the equity and of the debt will include also a risk premium directly linked to the premium earned over the risk free rate by the asset.

In greater detail, in a Black&Scholes (B&S) economy, the real world stochastic process followed by the assets A is:

$$dA_t = (r + \psi)A_t dt + \sigma A_t dW_t \tag{3}$$

where ψ is the parameter for the premium for market risk. Put in very simple terms, the value at time t the asset A is:

$$A(t) = \mathbf{E}\left[A(T)e^{-r(T-t)} - \int_t^T e^{-r(s-t)}\psi A_s ds\right] \tag{4}$$

Let the absolute present value of the market premium be $R_A = \int_t^T e^{-r(s-t)}\psi dA_s$. The expected return of the asset A is:

$$\mathbf{E}\left[\frac{A(T) - A(t)}{A(t)}\right] = \mathbf{E}\left[\frac{A(t)(e^{r(T-t)} - 1) + R_A e^{r(T-t)}}{A(t)}\right] = e^{r(T-t)} - 1 + \frac{\overline{R}_A e^{r(T-t)}}{A(t)} \tag{5}$$

where $\overline{R}_A = \mathbf{E}\left[\int_t^T e^{-r(s-t)}\psi dA_s\right]$. The return, once deducted the market risk premium, is obviously still the risk free one: $e^{r(T-t)} - 1$.

The terminal value of the equity is:

$$\mathbf{E}[E(T)] = \mathbf{E}\left[\max(A(T) - K, 0)\right] = \mathbf{E}^Q\left[\max(A((T) - K, 0)\right] + \overline{R}_E e^{r(T-t)} \tag{6}$$

where $\mathbf{E}^Q$ indicates that the expectation is taken under the risk-neutral measure and $\overline{R}_E = \mathbf{E}\left[R_E\right]$ is the premium for market risk, with:[1]

$$R_E = \int_t^T e^{-r(s-t)} A_s \frac{\partial E_s}{\partial A_s} \psi ds$$

The value at time t of the equity is the expected value discounted at the risk free rate (6), $E(t) = e^{-r(T-t)}\mathbf{E}\left[E(T) - R_E\right]$, which is the present value of the call on the asset A assuming it follows a risk neutral dynamics, minus the risk premium R_E, or equivalently $E(t) = e^{-r(T-t)}\mathbf{E}[E(T)] - \overline{R}_E$.

The return on the equity is:

$$\mathbf{E}\left[\frac{E(T) - E(t)}{E(t)}\right] = \frac{E(t)e^{r(T-t)} - E(t) + R_E e^{r(T-t)}}{E(t)} = e^{r(T-t)} - 1 + \frac{\overline{R}_E e^{r(T-t)}}{E(t)} \tag{7}$$

It is worth noting that the risk premium is variable and it depends on the financial leverage D/E: the higher it is, the greater the impact of the risk premium on the total return on the equity. Also in this case, the risk adjusted return is simply the quantity in equation (7) deducted the risk premium $\overline{R}_E e^{r(T-t)}/E(t)$, or the risk free return.

By exploiting the put-call parity for the default risk premium p the firm's debt is:

$$\begin{aligned} D(t) = e^{-r(T-t)}\mathbf{E}[D(T)] &= Ke^{-r(T-t)} + A(t) - Ke^{-r(T-t)} - E(t) \\ &= A(t) - E(t) \end{aligned} \tag{8}$$

which trivially confirms the balance sheet equivalence. From equations (5) and (7), the return on the debt is

$$\begin{aligned} \mathbf{E}\left[\frac{D(T) - D(t)}{D(t)}\right] &= \frac{A(t) - E(t)}{D(t)}(e^{r(T-t)} - 1) + \frac{(\overline{R}_A - \overline{R}_E)e^{r(T-t)}}{D(t)} \\ &= (e^{r(T-t)} - 1) + \frac{(\overline{R}_A - \overline{R}_E)e^{r(T-t)}}{D(t)} \end{aligned} \tag{9}$$

The WACC of the capital, when risk premiums are included, is equal to the return on the asset:

$$\begin{aligned} \frac{D(t)}{A(t)}\left[(e^{r(T-t)} - 1) + \frac{(\overline{R}_A - \overline{R}_E)e^{r(T-t)}}{D(t)}\right] + \frac{E(t)}{A(t)}\left[(e^{r(T-t)} - 1) + \frac{\overline{R}_E e^{r(T-t)}}{E(t)}\right] = \\ (e^{r(T-t)} - 1) + \frac{\overline{R}_A e^{r(T-t)}}{A(t)} \end{aligned} \tag{10}$$

The MM theorem is still valid also when the default of the firm is considered, as in Merton [5], but we work in a real world measure. The expected return on the capital structure

[1] See, for example, Merton [6], chapter 11.

(left-hand side in (10)) is equal to the expected return on the asset (right-hand side in (10)): in both cases it is equal to the risk free rate plus the market risk premium for the asset. The value of the firm is independent on the financial leverage and depends only on the value of the assets. The funding costs (and the **FVA**) are immaterial to evaluate assets, contracts or projects.

Assume now that the firm is able to invest in an asset earning more than the risk-free rate plus the premium for the market volatility risk: in a perfect market such asset does not exist, since there would be an arbitrage opportunity immediately cleared. In reality firms can exploit situations when they are able extract an extra-profit from their investments: this can be due to frictions and imperfections in the market, or to the ability of the firm to extract more value from its investment by the ability of its employees or by its more effective organisation compared with its competitors. A bank, for example, can earn extra-profits from its assets (*e.g.*: loans) for its superior knowledge of the creditworthiness of the clients/debtors with respect to other competing banks, so that it is able to create a so called *franchise*.

If such an asset exists, then its dynamics can be written as:

$$dA_t = (r + \psi + \pi)A_t dt + \sigma A_t dW_t \tag{11}$$

where π is t he parameter for the instantaneous extra-profit yielded by the asset. The value of the equity is computed by the B&S formula (2) with the argument of the Normal distribution functions modified as follows:

$$d_1 = \frac{\ln \frac{A_t}{K} + (r + \pi + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}; \quad d_2 = d_1 - \sigma\sqrt{T-t} \tag{12}$$

It is easy to check that the asset at time t is:

$$A(t) = \mathbf{E}\Big[A(T)e^{-r(T-t)} - \int_t^T e^{-r(s-t)}\pi A_s ds - \int_t^T e^{-r(s-t)}\psi A_s ds\Big] \tag{13}$$

The present value of the extra profit is $P_A = \int_t^T e^{-r(s-t)}\pi dA_s$, so that the expected return of the asset A is:

$$\mathbf{E}\Big[\frac{A(T)-A(t)}{A(t)}\Big] = \mathbf{E}\Big[\frac{A(t)(e^{r(T-t)}-1) + (P_A + R_A)e^{r(T-t)}}{A(t)}\Big] = e^{r(T-t)} - 1 + \frac{(\overline{P}_A + \overline{R}_A)e^{r(T-t)}}{A(t)} \tag{14}$$

where $\overline{P}_A = \mathbf{E}\left[\int_t^T e^{-r(s-t)}\pi dA_s\right]$. The return on the equity is accordingly modified and it is:

$$\mathbf{E}\left[\frac{E(T)-E(t)}{E(t)}\right] = e^{r(T-t)} - 1 + \frac{\overline{R}_E e^{r(T-t)}}{E(t)} + \frac{\overline{P}_E e^{r(T-t)}}{E(t)} \tag{15}$$

where $\frac{\overline{P}_E e^{r(T-t)}}{E(t)}$ is the expected extra-profit that the firm is able to create, with:

$$P_E = \int_t^T e^{-r(s-t)} A_s \frac{\partial E_s}{\partial A_s} \pi ds$$

The extra-profit grants a return over the risk adjusted return and it is a function of the financial leverage: the smaller the equity, the higher the extra-return over the market "fair" risk adjusted return.

It seems, at a first look, that if a firm is able to generate extra-profits, the optimal strategy for shareholders is to increase the financial leverage, *i.e.*: increase the D/E ratio, thus maximising the expected extra-return. This strategy is perfectly consistent with the risk preferences of the shareholders with respect to the market risk, since this is fairly remunerated by the market risk premium that the asset yields, and then more than fairly remunerated by the related market excess return over the risk free rate that the equity receives. Nonetheless, this strategy misses the point that the shareholders can be risk averse towards the firm's default event: the probability of such event increases as the leverage increases, but there is no theory on the extra-profit created by the firm and the remuneration of the default event risk borne by the shareholders.

Example 2.1. *Consider the data in table 1: the firm is able to invest in an asset that yields an extra-profit expressed as a continuous rate $\pi = 5\%$. The dynamics followed by the asset is given in equation* (11).

$A(t)$	100
r	5%
π	5%
σ	40%

Table 1: Data referring the asset A at time t.

In table 2 we show the value of the debt, of the equity, the present value of the expected extra-profit and extra-return generated by the firm for the shareholders. The extra-profits are the difference between the value of the equity valuated with formula (2)*, and the value computed with the same formula but the modified d_1 as shown in formula* (12).

Asset	Debt	Equity	D/E	Extra Profit	Extra Return	**PD** n.f	**PD**
100	47.24	52.76	0.90	5.1	9.58%	4.87%	3.73%
100	53.45	46.55	1.15	5.0	10.66%	9.17%	7.28%
100	59.33	40.67	1.46	4.8	11.87%	14.90%	12.19%
100	64.79	35.21	1.84	4.6	13.17%	21.73%	18.24%
100	69.77	30.23	2.31	4.4	14.55%	29.25%	25.11%
100	74.24	25.76	2.88	4.1	16.01%	37.02%	32.41%
100	78.19	21.81	3.59	3.8	17.51%	44.69%	39.80%
100	81.64	18.36	4.45	3.5	19.06%	51.99%	47.01%

Table 2: Value of the debt and of the equity, the present value of the expected extra-profit and extra-return, probability of default when the firm generates no extra-profits ($\pi = 0$, **PD** n.f) and when it does ($\pi = 5\%$, **PD**), for different levels of financial leverage

*From the table it can be observed that, on the one hand, the extra-return increases along with the leverage D/E; on the other hand, there is a higher and higher probability of default of the firm (**PD** n.f). It is worth stressing that the probability of default would be (even more) growing with the leverage also when no extra-profit is produced by the firm.*

In perfect efficient markets such extra-profit opportunities do not exist and the problem to choose the optimal capital structure that maximises the shareholders' return, given the remuneration of the default event risk, does not arise. Moreover, under the assumptions of efficient and frictionless markets, there is no need to consider the cost of the capital when deciding whether to invest it in a given asset: the cost of the equity and of the debt will be always such that it matches the yield of the asset. When it comes to the case of derivative contracts pricing, the results we have shown above support the argument of those that deem incorrect to consider the funding value adjustment (**FVA**, the funding costs paid on the debt needed to hedge the position), see amongst others Hull and White [3] and [4].

There are many assumptions that in practice invalidate the MM theorem (and the Merton model) on the irrelevance of the composition of the capital structure: for example, the costs related to the default imply that there is an optimal capital structure. We do not want to enter in this debate, though. Our aim is understanding if a justification of the inclusion of the funding costs (**FVA**), in the investment decisions of the firm, can be found within a classical, Merton-like, framework and which are the modification of the underlying assumptions needed to achieve it, in case.

There is one point that is overlooked in the analysis above of the Modigliani-Miller-Merton framework: while market agents can be risk averse and hence requiring a risk premium over the risk free return to compensate for market volatility, they are still neutral towards the event of firm's default. Actually in the one period Merton's setting, the default can happen only at the end of the only period that coincides with the end of the firm's activity anyway. In this framework, the default is simply a particular condition at the end of the activity, when the asset's value is not enough to completely pay back the debt and consequently the equity is worth nothing, given the limited responsibility of the shareholders.

In the end, in the Merton's model, the default indicates only those states of the world when the shareholders' investment in the firm's equity is completely lost, and the debtholders' investment in the firms' debt yields a negative return, possibly implying the total loss of the notional of the bond as well. There is no need to introduce a specific shareholder's risk aversion for the default in this model, since the premium for market risk is compensating for the market volatility the equity and, in the end, also the debt.

Reasons why the MM theorem, and the Merton's model results, do not hold in reality, may be found in the usually proposed following (not exhaustive) list:

- corporate taxes;
- bankruptcy costs;
- higher cost of raising external capital;
- managerial motives.

Without delving too much down into the possible conditions under which the capital structure does matter and, additionally, when shareholders are eager to maximise their experted profits taking into account also the default probability of the firm, we can state in very general terms that, in the real world, the managers of the firm run the business aiming at maximising the value of the equity (*i.e.*: the net firm's net value) under the constraint of an acceptable probability of default consistent with the shareholders' risk aversion to bankruptcy.

The second aspect that the classical theory of the firm does not consider, at least in the works of Modigliani and Miller [8] and Merton [5], is the multi-period duration of the business activity: at the end of the only period observed, the firm is not necessarily liquidated, on the contrary shareholders will very likely want to go on. When the activity's continuation is assumed, the analysis has to be accordingly extended.

3 Equity and Economic Capital

When shareholders are not indifferent to the firm's default event, then they are not indifferent either to the choice of the financial leverage, *i.e.*: the percentage of equity in the capital structure of the firm. Actually, strictly linked to the notion of equity, there is the notion of economic capital (EC), which is roughly defined as the amount of equity necessary to avoid the default of the firm at a given confidence level. For example, in the financial industry, national and international (see for example the Basel framework [9]) regulation forces the banks to have a given level of EC [2] to guarantee, over a time horizon typically of 1 year, the continuity of the business activity with a probability of 99%. This means that the shareholders (or, better, the regulators) accept 1% probability of bank's bankruptcy over 1 year. Alternatively said, EC is the minimum equity that can be used to finance an asset, together with the maximum amount of debt, subject to the constraint of a maximum acceptable probability of firm's default. Mangers seek to maximise profits requiring the minimum amount of equity from shareholders compatible with this constraint.

Assume we are working in a multi-period setting, with times $[t = T_0, T_1, T_2, ...]$ and let each sub-period $T_i - T_{i+1} = T$ be constant, so that $T_i - T_0 = iT$. We are at time t: the shareholders wish to invest the minimum quantity of equity necessary to keep enough EC so that at the end of the first sub-period $[T_0, T_1]$ the probability that the firm goes bankrupt is α.[3] To this end, given the distribution of all possible values $A(T_1)$, the firm computes the minimum level $A^{\alpha}_{T_0}(T_1)$ so that the probability to have a lower level is smaller than α, or:

$$A^{\alpha}_{T_0}(T_1) = A(T_1) : \{\mathbf{Pr}(A(T_1) \leq A^{\alpha}_{T_0}(T_1)) \leq \alpha\} \tag{16}$$

The quantity $\mathbf{VaR}^{1-\alpha}(A(T_1)) = \mathbf{E}[A(T_1)] - A^{\alpha}_{T_0}(T_1)$ is the Value-at-Risk of the asset at the $1-\alpha$ percentile: it represents the maximum unexpected loss occurring with a probability of $1-\alpha$. When the asset is log-normally distributed at a given time T_i, as implied by the

[2] The regulatory definition of EC is wider than the equity, including also some hybrid forms of capital.

[3] In some cases, such as in the financial industry, the minimum confidence level $1-\alpha$ (and hence the default probability α) is set by the regulation, so shareholders can only decides to have more a conservative level.

dynamics in (3) (setting $\psi = 0$), we get

$$A^{\alpha}_{T_0}(T_1) = A(T_0)e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon^{\alpha}} \tag{17}$$

where ϵ^{α} is is the point of the Normal standard distribution returning a probability $\mathbf{Pr}(A(T) \leq A^{\alpha}(T)) \leq \alpha$. As an example example, $\alpha = 1\%$ implies that $\epsilon^{\alpha} = -2.326$.

Let $M = e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon^{\alpha}}$, then for a generic time T_i we have that:

$$A^{\alpha}_{T_i}(T_{i+1}) = A(T_i)M$$

which is obviously stochastic. The expected value is:

$$\mathbf{E}\left[A^{\alpha}_{T_i}(T_{i+1})\right] = A(t)e^{r\times T\times i}M \tag{18}$$

If the firm wishes to exclude the possibility of default at $1-\alpha$ confidence level, at time T_0 it has to fund the asset A with a minimum amount of equity equal to the **VaR**:

$$\mathbf{EC}(T_1) = \mathbf{VaR}^{1-\alpha}(A(T_1))$$

At the end of the sub-period $[T_0, T_1]$, the EC covers the unexpected loss up to $\mathbf{EC}(T_1)$, so that expected terminal value minus the unexpected loss (at $1-\alpha$ c.l.) is the maximum notional amount of debt that can be used by the firm to fund the asset A:

$$K^{\alpha} = \mathbf{E}[A(T_1)] - \mathbf{VaR}^{1-\alpha}(A(T_1)) = \mathbf{E}[A(T_1)] - \mathbf{EC}(T_1) \tag{19}$$

At time T_0 the amount of equity should be the present value of the EC, or $E(T_0) = \mathbf{EC}(T_0) = e^{-rT}\mathbf{EC}(T_1)$. Nonetheless, it is quite easy to check that the equity needed to fund the asset is greater than the present value of the EC. In fact, since the default occurrence is not completely ruled out, the value of the debt at time t is:

$$D(T_0) = K^{\alpha}_{T_0}e^{-rT} - \mathbf{E}\left[\max[K^{\alpha}_{T_0} - A(T_1), 0]\right]e^{-rT} \tag{20}$$

so that, by balance sheet equivalence and from (19), the equity at time t is:

$$E(T_0) = A(T_0) - K^{\alpha}_{T_0}e^{-rT} + \mathbf{E}\left[\max[K^{\alpha}_{T_0} - A(T_1), 0]\right]e^{-rT} = e^{-rT}\mathbf{EC}(T_1) + p_{T_0} \tag{21}$$

where $p_{T_0} = \mathbf{E}\left[\max[K^{\alpha} - A(T_1), 0]e^{-rT}\right]$, is defined in section 2 as the compensation for expected credit losses requested by the debtholders. So, whichever is the chosen confidence level $1-\alpha$ by the shareholders, the equity needed to achieve this degree of protection against default is greater than the economic capital, since there is always a premium p_{T_0} requested by the debtholders, however small it may be (unless, obviously, shareholders willingly decide to finance the asset A with more equity).

In other words, the minimum level of equity to preserve the business continuity, at a given confidence level $1-\alpha$, is $\mathbf{EC}(t)$, but this level implies a present value of the debt that entails a level of the equity $E(T_0) = e^{-rT}\mathbf{EC}(T_1) + p_{T_0}$, greater than the budgeted amount $\mathbf{EC}(T_0)$ by the amount p_{T_0}. In practice, shareholders have to invest more than it is strictly required to avoid default at the end of the sub-period $[T_0, T_1]$ with probability α. From the shareholders' point of view, there is an excess equity invested to achieve a result in T_1 that would require a smaller amount.

Merton and Perold [7] define p_{T_0} as the *risk capital*, or the minimum amount of capital to run a limited liability company for one period. The EC we have defined above is the sum of the *risk capital* and the *cash capital* (still according to Merton and Perold [7]), or the amount of the equity above the minimum that the shareholders have to invest in the company.

It is worth stressing that p_{T_0} is the present value of the funding cost, defined in Castagna [1] (apart from the different notation) as $\mathbf{FC} = D^*(T_0) - D(T_0) = p_{T_0}$, or the present value of the extra interests, over the risk free ones, which the borrower (the firm in this case) has to pay to compensate the lender (the debtholders) for the expected losses on default. We have just shown above that the funding cost clashes with the extra equity over the economic capital, given the chosen level α, needed to run the business.

4 The Evolution of the (Net) Value of the Firm in a Multi-Period Setting

The shareholders are interested in a business activity that yields fair profits over a (possibly undefined) number of periods. The EC should preserve the life of the firm at a chosen confidence level, but it is important to model the evolution of the net value of the firm over the future sub-periods, to assess the conditions under which the business activity yields a fair return on the investment.

The notion of (net) value of the firm is similar to that of the equity, but it allows to analyse the ability of the firm to be profitable and to survive over time. Actually the value of the firm $\mathbf{VB}(T_i)$, at a generic time T_i, is simply the difference between the value of the asset $A(T_i)$ and the notional amount of the debt K_{T_i}; this means that the value of the firm can be also a negative number. The relationship between the equity $E(T_i)$ and the value of the firm $\mathbf{VB}(T_i)$ is obviously:

$$E(T_i) = \max[\mathbf{VB}(T_i), 0]$$

We examine how $\mathbf{VB}$ evolves over time.

From equation (21) we know which is the level of the equity at time $t = T_0$, $E(t) = E(T_0)$. At the end of the first period $[T_0, T_1]$, the value of the firm (for the shareholder) is:

$$\mathbf{VB}(T_1) = E(T_0) + [A_{T_1} - K^{\alpha}_{T_0} - E(T_0)] \tag{22}$$

Equation (22) states that the value of the firm at time T_1 is equal to the sum of the equity at time $t = T_0$, $E(T_0)$, plus the profits (or losses) generated by the investment in asset, whose value is $A(T_1)$, net of the repayment of the debt K^{α} and the initial investment $E(T_0)$. To pay back the expired debt, the firm sells the asset and uses the proceeds. An alternative option, more viable in practice, is to roll over the debt by resorting to a new debt, which we will assume to be maturing in one period as the first one. The roll-over of the debt implies a continuity of the activity that we will shortly introduce.

$\mathbf{VB}(T_1)$ is a stochastic variable and the equity is equal to it when positive: when it goes below zero, the equity is floored at this level, by the limited liability of the shareholders, and the firm is declared bankrupt. We want to ascertain if and when the expected value of the firm breaches the zero lower boundary; in other words we want to investigate if on average the business activity generates profits or losses, and in which cases the firm is doomed to default after a given number of years.

The expected value of (22) is:

$$\mathbf{E}[\mathbf{VB}(T_1)] = E(T_0) + [A(T_0)e^{rT} - K^{\alpha}_{T_0} - E(T_0)] = A(T_0)e^{rT} - K^{\alpha}_{T_0} \qquad (23)$$

We would like to stress that $\mathbf{E}[\mathbf{VB}(T_1)]/E(T_0) - 1$ is not the expected return of the investment in the equity in the first period, which we know to be the risk free rate, when considering also the limited liability.[4]

In T_1 the firm does not stop the activity and it wants to go on investing in A, so it wants to immediately buy it back: to do so it must borrow money by new debt. The buy back is not needed when the new debt is used to directly pay back the old maturing debt, thus avoiding the sale of the asset. Shareholders do not change their preferences towards the default event risk, so they still want to be sure that there is an EC covering $1-\alpha$ of unexpected losses. The expected EC in T_1 can be easily derived by means of equation (18):

$$\mathbf{E}[\mathbf{EC}(T_2)] = \mathbf{VaR}^{1-\alpha}(A(T_2)) = \mathbf{E}[A(T_2)] - A^{\alpha}_{T_1}(T_2) = A(T_0)e^{r\times T\times 2} - A(T_0)e^{r\times T}M$$

The expected level of the debt consistent with this level of EC is

$$K^{\alpha}_{T_1} = A(T_0)e^{r\times T}M = K^{\alpha}_{T_0}e^{rT}$$

The expected default premium p_{T_1} can be also easily derived as:

$$\mathbf{E}[p_{T_1}] = \mathbf{P}(\mathbf{E}[A_{T_1}], K^{\alpha}_{T_0}e^{rT}, T, \sigma) = A(T_0)e^{rT}\mathbf{P}(1, M, T, \sigma)$$

where $\mathbf{P}(1, M, T, \sigma)$ is the value of a put option according to the Black&Scholes model[5] when the underlying spot price is 1, the strike is M and the time to maturity is T (which is the length of each sub-period) and the volatility is σ. In our setting it is straightforward to check that $\mathbf{P}(1, M, T, \sigma)$ is a constant. In last part of the equation above we have exploited the homogeneity of the Black&Scholes's formula. We can note that for the first period $[T_0, T_1]$, the credit premium is:

$$p_{T_0} = A(T_0)\mathbf{P}(1, M, T, \sigma)$$

so that we can rewrite the expected credit premium as:

$$\mathbf{E}[p_{T_1}] = A(T_0)e^{rT}\mathbf{P}(1, M, T, \sigma) = e^{rT}p_{T_0}$$

The expected value of the debt in T_1, expiring in T_2, is then:

$$\mathbf{E}\left[D(T_1)\right] = K^{\alpha}_{T_1}e^{-rT} - A(T_0)e^{rT}\mathbf{P}(1, M, T, \sigma) = K^{\alpha}_{T_0} - e^{rT}p_{T_0} = D(T_0)e^{rT}$$

[4] Recall we have set the market risk premium $\psi = 0$.

[5] See above, where we used the put-call parity to derive the value of the credit premium/put option.

Combining these results, we have that the expected equity needed to carry on the activity for the following period $[T_1, T_2]$ is:

$$\mathbf{E}\left[E(T_1)\right] = A(T_0)e^{rT} - K^{\alpha}_{T_0} + e^{rT}p_{T_0} = E(T_0)e^{rT} \tag{24}$$

Equation (24) shows that the expected equity needed to carry on the business activity of the firm grows at the risk free rate. Nonetheless, the expected value of the firm at the end of the first period in T_1 is the quantity in (23), so that the net imbalance is:

$$\mathbf{E}\left[\mathbf{VB}(T_1) - E(T_1)\right] = -e^{rT}p_{T_0} \tag{25}$$

which shows that, once the roll-over of the debt and the re-investment in the asset is operated, the firm will loose an amount equal to the funding costs, since, as we have stressed before, p_{T_0} is the present value of the funding costs.

The firm can now face two alternative that we analyse separately.

4.1 Increase of the Financial Leverage

The firm accepts the fact that the maximum expected equity that can be used is equal to $\mathbf{VB}(T_1)$, so that the only way to re-invest in asset $A(T_1)$ is to increase the amount of debt. The expected notional amount of the debt needed at time T_1 can be in general terms defined as:

$$K_{T_1} = K : \{\mathbf{E}\left[A(T_1) - D(T_1)\right] = \mathbf{E}\left[\mathbf{VB}(T_1)\right]\} \tag{26}$$

This amount will be greater than $K^{\alpha}_{T_1}$, since $\mathbf{E}\left[\mathbf{VB}(T_1)\right] < \mathbf{E}\left[E(T_1)\right]$, and the probability of default of the firm will increase at each time,

$$\mathbf{Pr}\left(\mathbf{E}\left[A(T_i) \leq K_{T_i}\right]\right) > \mathbf{Pr}\left(\mathbf{E}\left[A(T_{i-1}) \leq K_{T_{i-1}}\right]\right)$$

and it will approach 1, or the default will be certain. In fact, the general form of equation (23):

$$\mathbf{E}[\mathbf{VB}(T_i)] = A(T_0)e^{r\times T\times i} - K_{T_i}$$

where $K_{T_i} > K_{T_{i-1}}$. Since $\ln(K_{T_i}/K_{T_{i-1}}) > \ln(A_{T_i}/A_{T_{i-1}})$, then the firm is expected to stop its activity at the time T_n for which $K_{T_n} \geq A(T_0)e^{r\times T\times n}$: technically it is a default.

The strategy to increase the amount of debt to keep the expected amount of invested in asset A unchanged implies that the shareholders are, after each period, modifying their risk aversion towards the firm's default event. The economic capital is able to cover unexpected losses at gradually declining confidence levels, until the bankruptcy is a sure event.

Moreover, the return on equity will not be the fair risk free rate (plus the market risk premium when it is different from zero), so the strategy to keep the amount invested in the asset constant seems to be inconsistent with the starting assumptions on the firm's default risk aversion of the shareholders and the ability to produce fair profits over time.

Example 4.1. *Consider the data referring to the asset $A(T_0)$ in table 1: for simplicity we assume that the firm is not able to produce any extra-profit, whence $\pi = 0$, and that there is no risk premium for market risk, $\psi = 0$; furthermore, $T_i - T_{i-1} = T = 1$. The* *shareholders are risk averse to default and they wish that the probability that the firm goes bust is lower than $\alpha = 1\%$, so that the firm survives at each period T_i with a confidence level of $1 - \alpha = 99\%$.*

In table 3 we show what happens to the value of the firm and the expected profits/losses generated by the business activity for the first five years (the results are qualitatively the same for all next periods) . At time $t = T_0 = 0$, the situation is not different from the standard Modigliani-Miller-Merton framework: Given the acceptable $\mathbf{PD} = 1\%$, the debt notional is consistently set at 64.71, implying a debt value of 61.51 (the default premium is 0.0039) and the equity is 38.49.

At time $T_1 = 1$, to keep the constant the investment is the asset A, whose expected value is $A(T_1) = 105.13$, and the $\mathbf{PD} = 1\%$. the firm has to set the notional of the debt at 68.03, implying a debt value of 64.67. The equity needs an equity of 40.46, but the value of the firm grew only up to 40.42.

The firm decides to increase the notional of the debt up to a level which allows the reinvestment in the asset, with an equity equal to the expected value of the firm: this level is recursively computed and it is 68.07, implying a debt value of 64.71 that summed to 40.42 yields the expected asset value 105.13. This procedure is repeated for all subsequent periods.

The last three columns show that this strategy on the one hand increases the debt/equity ratio and the probability of default of the firm, hence it is not consistent with the shareholders' risk aversion towards default. On the other hand, the strategy is incapable to generate a return enough to match the risk free return (5% in this example). In fact, the return slowly declines and it is always below the risk free rate.

It is also worth of note that the (continuous) rate at which the notional increases is greater than the rate at which the asset is expected to grow, 5%, which means that the strategy is unsustainable in the long run and a default is expected to surely happen, although in a very large number of years given the low $\mathbf{PD}$ *chosen at the inception of the firm.*

T_i	$\mathbf{E}[A(T_i)]$	$\mathbf{E}[K^{\alpha}_{T_i}]$	$\mathbf{E}[p_{T_i}]$	$\mathbf{E}[D(T_i)]$	$\mathbf{E}[E(T_i)]$	$\mathbf{E}[\mathbf{VB}(T_i)]$	K_{T_i}	$D(t)$	K/E	$\mathbf{E}[PD(T_i)]$	ROE
0	100.00	64.71	0.039	61.51	38.49	38.49	64.71	61.51	1.68	1.00%	
1	105.13	68.03	0.041	64.67	40.46	40.42	68.07	64.71	1.68	1.01%	4.8980%
2	110.52	71.51	0.043	67.98	42.53	42.45	71.61	68.07	1.69	1.02%	4.8968%
3	116.18	75.18	0.046	71.47	44.71	44.58	75.33	71.61	1.69	1.03%	4.8959%
4	122.14	79.04	0.048	75.13	47.01	46.81	79.24	75.33	1.69	1.04%	4.8945%
5	128.40	83.09	0.050	78.99	49.42	49.16	83.36	79.24	1.70	1.04%	4.8935%

Table 3: Columns from two to five show, for each period, the expected asset value, the expected notional and value of the debt, and the expected equity needed to keep the probability of default at 1%. Columns from 6 to 12 show the evolution of the expected value of the firm and the consequent notional, default premium and value of the debt, the debt notional/equity ratio, the probability of default and the return on equity.

4.2 Shrinking of the Balance Sheet

A second strategy that can be pursued complies with the risk aversion of the shareholders, made explicit as the acceptance of the occurrence of the bankruptcy with a probability equal or below a chosen level α. This strategy recognises that the only way the keep the firm's probability of default at the level set at time T_0 is to shrink the balance sheet, accordingly to the expected value of the firm in T_1.

In greater detail, by equations (24) and (27), we can write the expected value of the firm in T_1 as:

$$\mathbf{E}\left[\mathbf{VB}(T_1)\right] = [E(T_0) - p_{T_0}]e^{rT} = \beta_{T_1}E(T_0)e^{rT} \tag{27}$$

where

$$\beta_{T_1} = \left(1 - \frac{p_{T_0}}{E(T_0)}\right)$$

The factor $\beta_{T_1} < 1$ and it represents also the smaller amount to invest in asset $A(T_1)$, consistent with an expected economic capital $\mathbf{E}\left[\mathbf{EC}(T_1)\right]$ and a related expected equity $\mathbf{E}\left[E(T_1)\right] = \mathbf{E}\left[\mathbf{VB}(T_1)\right]$ needed to carry on the business activity, still keeping the probability of default $\mathbf{Pr}\left(\mathbf{E}\left[A(T_2) \leq K_(T_2)\right]\right) \leq \alpha$. In fact, the balance sheet equivalence will read as follows:

$$\beta_{T_1}\mathbf{E}\left[A(T_1)\right] = \beta_{T_1}\mathbf{E}\left[D(T_1) + E(T_1)\right]$$

which is exactly the same equivalence of the case of an investment in $A(T_1)$, times the shrinking factor β_{T_1}.

Also in this case it is possible to derive a general form of equation (23):

$$\mathbf{E}[\mathbf{VB}(T_i)] = \beta_{T_i}[A(t)e^{r\times T\times i} - K_{T_i}]$$

where

$$\beta_{T_i} = \left(1 - \frac{p_{T_0} \times i}{E(T_0)}\right)$$

The factor $\beta_{T_i} < \beta_{T_{i-1}}$, and it will reach a lower boundary at zero after a given number of periods n, when the expected firm's value will be zero as well and the default will be certain, even though at each time the default probability is still kept equal to or lower than α: $\mathbf{Pr}\left(\mathbf{E}\left[A(T_i) \leq K_{T_i}\right]\right) \leq \alpha$

In conclusion, also the strategy to shrink the balance sheet, although consistent with the shareholders risk aversion to firm's default, leads to a sure arrest of the firm's operations on an expectation basis, without mentioning the fact that on average the profits produced are not sufficient to grant a return on equity equal to the risk free return.

Example 4.2. *Let us revert to the example 4.1: after the first period, the firm decides to invest in the asset A for quantity smaller than the initial one, according to the factor β. In table 5 we show how the expected value of the firm evolves when the balance sheet is shrunk according to the factor β: it is obvious that the value of the firm and of the equity are equal by construction of the strategy.*

The debt/equity ratio and the **PD** *keep constant at the initial levels for all the years considered. Nonetheless, the expected return on equity is below the risk free rate; since the β constantly declines, in the long run the shrinking of the balance sheet will reduce the equity to zero: with the starting* $\mathbf{PD} = 1\%$, *β is expected to drop to zero after 980 periods.*

Table 4: Add caption

T_i	$\mathbf{E}[\beta_{T_i} A(T_i)]$	$\mathbf{E}[\beta_{T_i} K_{T_i}]$	$\mathbf{E}[\beta_{T_i} D(T_i)]$	$\mathbf{E}[\beta_{T_i} E(T_i)]$	$\mathbf{E}[\mathbf{VB}(T_i)]$	β_{T_i}	D/E	$\mathbf{E}[PD(T_i)]$	ROE
0	100.00	64.71	61.51	38.49	38.49	1.0000	1.68	1.00%	
1	105.02	67.96	64.60	40.42	40.42	0.9990	1.68	1.00%	4.8980%
2	110.29	71.37	67.84	42.45	42.45	0.9980	1.68	1.00%	4.8978%
3	115.83	74.95	71.25	44.58	44.58	0.9969	1.68	1.00%	4.8977%
4	121.64	78.71	74.83	46.82	46.82	0.9959	1.68	1.00%	4.8976%
5	127.75	82.66	78.58	49.17	49.17	0.9949	1.68	1.00%	4.8975%

Table 5: Columns 5 and 6 show the evolution of the expected value of the firm and the factor β for each period. Columns 2 to 4 show the expected values of the amount invested in the asset, the expected notional and value of the debt and the expected value of the equity. The debt notional/equity ratio, the probability of default and the return on equity are in the last three columns.

5 Charging Funding Costs and FVA

In the previous section we have analisyed how the net value of the firm evolves over time: following the expected imbalance (loss) at the end of the first period $[T_0, T_1]$, the firm can operate two strategies. The first one keeps the expected amount invested in the asset A constant, but it gradually increases the firm's probability of default because the leverage D/E increases in each period. The second strategy keeps constant the probability of default, but the amount invested in the asset declines in each period by a factor β. The expected final result of both strategies is that the firm produces not enough profits on average and the end of activity is certain on an expectation basis after a given number of periods.

The only way for the firm to generate on average a fair profit for the equity and to continue the business activity without shrinking the balance sheet, but still preserving its surviving at the confidence level chosen at the inception T_0, is to make sure that the asset A not only grows at an expected rate equal to the risk free rate,[6] but it generates a return that covers also the funding cost. This is clear from the inspection of equations (24) and (27).

Actually, if the asset A yields, at the end of each period $[T_i, T_{i+1}]$, not only the risk free rate (plus the market risk premium if different form zero), but also an additional amount equal to $e^{rT} p_{T_i}$, then equation (24) reads, in its general form:

$$\mathbf{E}[\mathbf{VB}(T_i)] = A(t)e^{r\times T\times i} + e^{r\times T\times i} p_{T_{i-1}} - K^{\alpha}_{T_0} e^{r\times T\times (i-1)} \tag{28}$$

and then (27) is:

$$\mathbf{E}\left[\mathbf{VB}(T_i) - E(T_i)\right] = 0 \tag{29}$$

which means that, at the end of each period, the firm can carry on the activity by reinvesting in the asset without any increase of the debt/equity ratio, and hence higher probabilities of default, or shrinking of the balance sheet.

In other words, the firm should seek to earn from the asset not only the fair return, but only an extra-yield such that the funding costs are covered, given the equivalence

[6] If the market risk premium is not set equal to zero, the expected growth rate includes this too, clearly.

between these and $p_{T_{i-1}}$ which we have pointed out before. This means also that the firm needs to invest in assets for which the extra-profit rate $\pi > 0$.

If the firm invests the capital in assets whose price is completely out of its control (*e.g.*: they are traded in perfect and frictionless markets), then there is little it can do and the shareholders must accept the fact that the expected value of the firm will not grow enough to grant a fair return.

When, on the contrary, the firm has some bargaining power in setting the value of the asset A, then the shareholders can demand from the firm's management to wield it and charge the asset's value with the present value of funding costs p.

We can then conclude that the inclusion of funding costs into the pricing of the asset is fully justified even in a Modigliani-Miller-Merton framework, when we extend the analysis in a multi-period setting. It is also worth noting that the analysis we have sketched also fully justifies, as a robust and theoretically sound practice, the use of risk adjusted measures to price and assess the profitability of the assets.

For example, the RAROC (Risk Adjusted Return on Capital), is defined as:

$$\mathbf{RAROC}(T_i) = \frac{\mathbf{PL}}{\mathbf{EC}(T_{i-1})} \tag{30}$$

where

$$\begin{aligned}\mathbf{PL} &= \mathbf{E}[\mathbf{VB}(T_i) - \mathbf{EC}(T_{i-1})] \\ &= [A(t)e^{r\times T\times i} - K^{\alpha}_{T_0}e^{r\times T\times(i-1)}] - [A(t)e^{r\times T\times(i-1)} - e^{-r\times T}K^{\alpha}_{T_0}e^{r\times T\times(i-1)} + e^{-r\times T}p_{T_{i-1}}]\end{aligned} \tag{31}$$

when the asset yields the risk free rate only.

The RAROC should equal the expected fair return on equity (the risk free rate plus risk premium). In our setting, with the market risk premium equal to zero, the RAROC is equal to the fair risk free rate return if the asset yields also an additional quantity equal to the funding costs, $p_{T_{i-1}}$.

Remark 5.1. *The result we have reached implicitly hinges on the idea that the firm does not consider its default when computing the evolution of its expected value and equity needed to carry on the business activity. Ex ante, the balance between the expected value and the expected equity needed is prevented to be negative: but this means that the firm is excluding the possibility of going bust, although it is actually accepting the occurrence of this event with a probability below* α.

This means that, ex post, given the fact that the firm will survive, it will earn an extra profit that will never be lost in the future, even if the firm bankrupt, since in this case the limited liability will never expose the shareholders to the negative part of the $\mathbf{PL} = \mathbf{E}[\mathbf{VB}(T_i) - \mathbf{EC}(T_{i-1})]$.

In conclusion, on one hand the firm sets the desired return on its investment at a level consistent with $\mathbf{E}[\mathbf{VB}(T_i) - \mathbf{EC}(T_{i-1})] \geq 0$*, so as to avoid a trend which will lead to the default, but on the other hand, once the default does not occur, the actual profit earned by the shareholders will be above the fair rate. In the simple extension of the MM and Merton setting we have introduced above, which is still too far from the actual working of the real business activity, we are not able to exclude this aporetic aspect of the result.*

6 Conclusion

By extending the Modigliani-Miller-Merton framework to a multi-period setting, where the shareholders are not indifferent to the default of the firm, we obtain two results:

- the capital structure is no more irrelevant (MM theorem does not hold), since the equity, seen as EC, is set at an optimal level such that it keeps the default probability below a threshold α;
- the inclusion of funding costs, and of the **FVA** for derivative contracts, in the evaluation process is fully justified because the firm needs to recover their amount, beside the fair return yielded by the assets, to grant a sustainable and profitable continuity of the business activity, still keeping the probability of default below α.

The theoretical result to exclude funding costs from the valuation of assets and contracts holds only in a single period setting in which the firm's default event is immaterial to shareholders.

The result still does not lie on a firm ground, since the general framework we worked in is too far from the real functioning of the business activity. In particular, not only the indifference to default of the shareholders and the single period setting must be removed (as we did in the present work), but also the possibility to have a whole range of available investments, possibly funded with rolling liabilities, has to be considered. This is important especially in some industries such as the banking and financial services. We leave this further refinement of the theory to future research.

References

[1] A. Castagna. Funding, liquidity, credit and counterparty risk: Links and implications. *Iason research paper. Available at http://iasonltd.com/resources.php*, 2011.

[2] A. Castagna. Yes, FVA is a cost for derivatives desks. A note on Is FVA a cost for derivatives desks? by Prof. Hull and Prof. White. *Iason research paper. Available at http://iasonltd.com/resources.php*, 2012.

[3] J. Hull and A. White. Is fva a cost for derivatives desks? *Risk*, (July):83–85, 2012.

[4] J. Hull and A. White. Collateral and credit issues in derivatives pricing. *Working Paper*, 2013.

[5] R. Merton. On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, (2):449–471, 1974.

[6] R.C. Merton. *Continuous-Time Finance*. Basil Blackwell Inc., 1990.

[7] R.C. Merton and A.F. Perold. Theory of risk capital in financial firms. *Journal of Applied Corporate Finance*, 6(3):16–32, 1993.

[8] F. Modigliani and M.H. Miller. The cost of capital, corporation finance and the theory of investment. *The American Economic Review*, 48(3):261–297, 1958.

[9] Basel Committee on Banking Supervision. Basel ii: International convergence of capital measurement and capital standards: A revised framework. *Available at http://www.bis.org*, (June), 2006.