THE MAIN RISKS OF AN FX OPTIONS' PORTFOLIO

ABSTRACT. In this document we study the risks of an FX options' portfolio; we will focus on which, we think, are the main sources of risk. Besides, we analyze also the relationships between the underlying assets' price and the (possibly stochastic) volatility. Some practical suggestions will be proposed to calculate the a total (combined asset's price-volatility) VAR of a portfolio.

1. INTRODUCTION

In this work we identify and study the main risks' sources of a portfolio of options, that is: the underlying asset's price and the volatility. We focus on options written on currencies, but results apply to almost any kind of options, though specific issues should be analyzed and taken into account according to the different cases.

We use as a benchmark the well known Black and Sholed (1973) (hereon, B&S) model; the formula to price an option at time t and price of the underlying equal to S, with strike K, expiry at T, is:

$$\mathcal{O}(S,t,T,K,\sigma_K,r_d,r_e,\omega) = Df^d(t,T)\left[\omega F(t,T)\Phi(\omega d_1) - \omega K\Phi(\omega d_2)\right] \tag{1}$$

where $\Phi(x)$ is the cumulative distribution function of standardized Normal variable evaluated in x, and

$$d_1 = \frac{\ln\frac{F(t,T)}{K} + \frac{\sigma_K^2(t,T)}{2}(T-t)}{\sigma_K(t,T)\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma_K(t,T)\sqrt{T-t}$$

The formula has been written in terms of the forward price of the underlying asset:

$$F(t,T) = S_t\frac{Df^d(t,T)}{Df^d(t,T)}$$

where the discount factors Df^d and Df^e are defined as:

$$Df^d = e^{-\int_t^T r^d(t)dt}$$

$$Df^e = e^{-\int_t^T r^f(t)dt}$$

$r^d(t)$ and $r^f(t)$ are, respectively, the instantaneous domestic and the external risk-free rate. The parameter ω is set equal to 1 if we want to price a call option, whereas is is set equal to -1 in case a put option has to be priced. Besides we use the following notation:

$$\phi(x) = \Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

The option price is the solution of the PDE:

$$\frac{\partial\mathcal{O}}{\partial t} + \frac{1}{2}S^2\sigma_K^2\frac{\partial^2\mathcal{O}}{\partial S^2} + (r^d(t) - r^f(t))S\frac{\partial\mathcal{O}}{\partial S} = r^d(t)\mathcal{O} \tag{2}$$

provided the proper boundary and terminal conditions.

It is convenient to provide the derivatives of the B&S with respect to variables and parameters.

$$\Delta(S,t,T,K,\sigma_K,r^d,r^f,\omega) = \frac{\partial\mathcal{O}}{\partial S} = \omega Df^f(t,T)\Phi(\omega d_1)$$

$$\Gamma(S,t,T,K,\sigma_K,r^d,r^f,\omega) = \frac{\partial^2\mathcal{O}}{\partial S^2} = \frac{\omega Df^f(t,T)\phi(\omega d_1)}{S\sigma_K(t,T)\sqrt{T-t}}$$

$$\Theta(S,t,T,K,\sigma_K,r^d,r^f,\omega) = \frac{\partial \mathcal{O}}{\partial t} = \frac{\omega Df^f(t,T)\phi(\omega d_1)S\sigma_K(t,T)}{2\sqrt{T-t}} + \omega Df^f(t,T)r^f(t)SN(\omega d_1) - \omega Df^d(t,T)r^d(t)KN(\omega d_2)$$

$$\mathcal{V}(S,t,T,K,\sigma_K,r^d,r^f,\omega) = \frac{\partial \mathcal{O}}{\partial \sigma_K} = Df^f(t,T)S\sqrt{T-t}\phi(\omega d_1)$$

$$\mathcal{W}(S,t,T,K,\sigma_K,r^d,r^f,\omega) = \frac{\partial^2 \mathcal{O}}{\partial \sigma_K^2} = \frac{Df^f(t,T)S\phi(d_1)d_1d_2\sqrt{T-t}}{\sigma_K(t,T)}$$

$$\mathcal{X}(S,t,T,K,\sigma_K,r^d,r^f,\omega) = \frac{\partial^2 \mathcal{O}}{\partial \sigma_K \partial S} = \frac{-Df^f(t,T)\phi(d_1)d_2}{\sigma_K(t,T)}$$

The formulae are self-explicatory and we refer to Hull (2003) for a more general treatment of the pricing formula and its derivatives. We would like to spend a few words on the last three derivatives. The first one, $\mathcal{V}$, is the vega of the option and it is its sensitivity to changes of the implied volatility of the underlying price process, σ. The $\mathcal{X}$, usually called vanna, measures the variation of the $\mathcal{V}$ due to a change of the underlying asset price; the $\mathcal{W}$, usually called volga, measures the variation of the $\mathcal{V}$ due to a change in the implied volatility parameter, σ.

2. Underlying Asset's Price Risk

The first and more evident risk arising form taking a position in options is given by the movements in the underlying asset. It is very easy to get a good picture of this risk for a single option but it is a very tricky task to define a sensible value-at-risk for a complex portfolio of option over a given period of time. Let's start for a portfolio Π of a single option $\mathcal{O}$; we can use a Δ-Γ approximation to infer the risk due a movement in the underlying asset's price associated to this position:

$$d\Pi = \Delta dS + \frac{1}{2}\Gamma dS^2 + \epsilon \tag{3}$$

where ε is the approximation error. If we assume that the underlying assets's price returns dS/S are normally distributed and that the variance is V over the period we chose to monitor the value at risk, then it is well known that the 99% confidence level of the price variation is $\overline{dS} = 2.33$S $\sqrt{V}$ in absolute terms; so if we replace dS with $\overline{dS}$ in (3) we get the 99% value at risk of the single option portfolio:

$$\overline{d\Pi} = \Delta\overline{dS} + \frac{1}{2}\Gamma\overline{dS}^2 + \epsilon$$

This approximation is rather good for small changes of the underling S and short time horizons, but its accuracy declines very rapidly with increasing volatility of the underlying asset and time horizons. As an example, in figure 1 we plot the value of a call option priced via (1) for different levels of the asset's price, and the corresponding Δ-Γ approximation provided in (3). The option is a call expiring in 3 months, written on the Eur/Usd exchange rate, whose price is 1.2950, and struck at 1.2950. The implied volatility used in the B&S formula is set at 8.6%. It is rather clear that 3 over-(under-)estimates the VaR of one option's long (short) position. But in general we can still assume that it is a good approximation, at least for small change in the underlying.

The Δ-Γ approximation dramatically worsens when we consider complex options portfolios including thousands of options (and this is the case of any market maker) and it becomes is meaningless, if not misleading at all. We show that with a very simple portfolio containing a $\mathcal{V}$-weighted call spread written on the Eur/Usd exchange rate (whose price is again 1.2950); the expiry is still in 3 months, and the strike is 1.2950 for the first option, which is bought in 1 unit, and 1.3500 for the second option which is sold in 1.27 units. The quantity are chosen so to make the portfolio $\mathcal{V}$-neutral: if the implied volatility (which is set again at 8.6%) change, the value of the portfolio is not affected, everything else being equal. It is quite clear that the approximation is good only for a very little changes in the underlying asset's price, whereas it is completely deteriorated for reasonable (at least for VAR purposes) changes. The problem mainly resides in the fact that at the starting point the Γ of the portfolio is 0, since it is a $\mathcal{V}$-weighted spread, and the approximating function is a rect line; moreover, the portfolio is not a monotonic function,

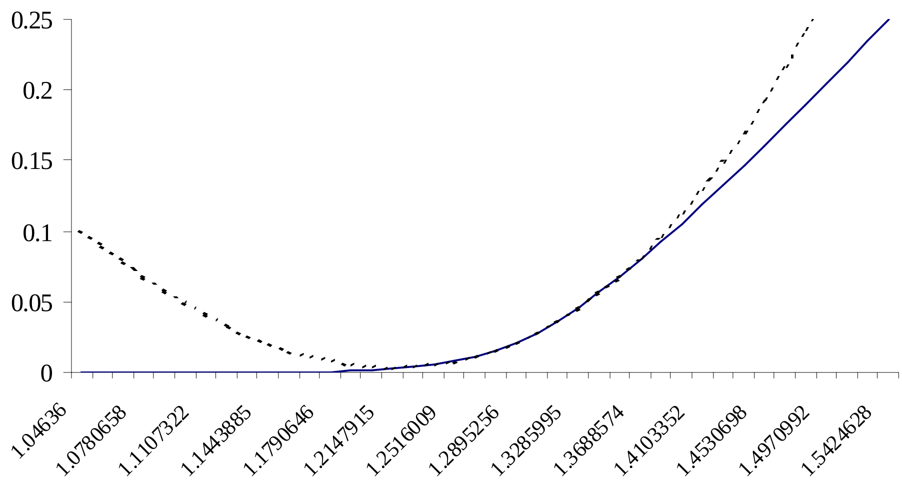

FIGURE 1. Δ-Γ approximation of a call option; x-axis: underlying asset's price, y-axis: portfolio's value in Usd

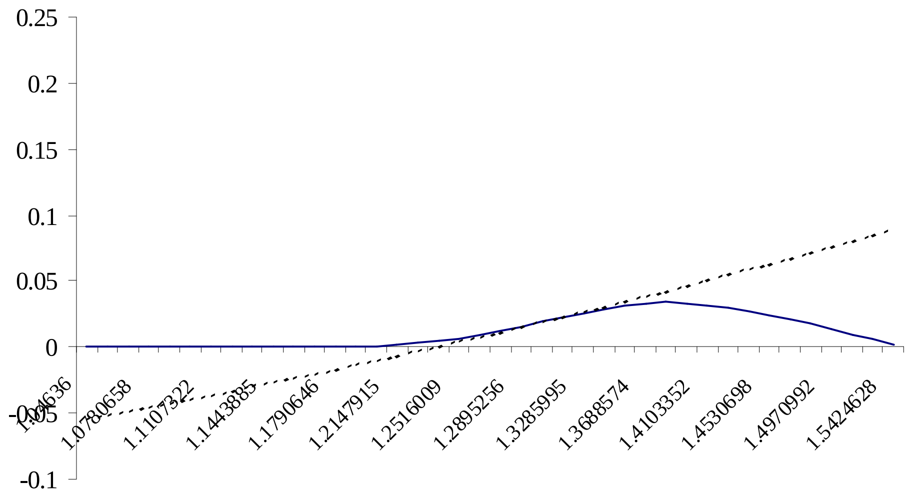

FIGURE 2. Δ-Γ approximation of a $\mathcal{V}$-weighted call spread; x-axis: underlying asset's price, y-axis: portfolio's value in Usd

and this hinders a good approximation by a second order function like the Δ-Γ one. Besides, the worst scenario for the value of the portfolio is for large increase of the underlying asset's price, absolutely wrongly predicted by the approximation, which instead over-estimates the risk on the downside. If we had sold a $\mathcal{V}$-weighted call spread, then it is easy to see that the worst case is given by a moderate increase of the underlying asset's price; whereas a large movement would produce a profit. The situation is still worse for portfolios including many options with different strikes and different maturities.

The example above should make clear that the Δ-Γ approximation has to dropped and another estimation of the underlying risk must be adopted. We suggest one possible solution, given by the following procedure:

- after having forecasted the volatility of the underlying assets' price, determine the relevant variation $\overline{dS}$ at the desired confidence level;
- divide the range $R = 2\overline{dS}$ in $N = int(\frac{R}{\delta S})$ intervals, large δS;
- run step by step the range of prices at δS pace and revaluate at each level $S_i = S_0 + i\delta S$, with $i = \{0, ..., N\}$, obtaining the portfolio's value $\Pi_i = \Pi(S_i)$;
- the VAR for the chosen period, and referred to the underlying risk, is then $\max_{i=1,...,N}(\delta\Pi_i^S)$, where $\delta\Pi_i^S$ is the variation in value of the portfolio between the current level of the asset's price S and each S_i.

In this section we did not studied how to forecast the volatility of the underlying asset: a wide range of techniques are available to estimate this quantity, though one is making more or less strong assumptions on the properties of the price distribution (both terminal and transition one). Many articles and books are devoted to this task and we refer to them for further analysis of this issue, since we deem it is beyond the scope of this study to investigate it.

We summarize the results of this section in the following

Facts 2.1.
(1) the VAR of a single option can be quite well captured by a Δ-Γ approximation; this is in general true for very simple portfolios whose P&L profile behaves like a monotonic function;
(2) complex portfolios, whose P&L behaves like a non-monotonic function, cannot be captured by Δ-Γ approximation, which can be meaningless for sensible movements of the underlying asset; this is almost always the case in the real world;
(3) the maximum VAR of complex portfolios is not necessarily given by large movements of the underlying asset's price;
(4) the only way to estimate correctly the VAR is to revaluate the portfolio for different levels of price of the underlying asset, within the range predicted at a given confidence level.

3. Volatility Risk: Hedging with B&S and Fixed Implied Volatility

We extend our analysis tho the second main source of risk of an option position: the volatility. This is also the main risk in a book of options of a market maker, or a volatility trader in general: these operators are generally Δ-hedged, so that the risk arising from the exposure to the movements of the underlying asset is very limited. Hence, the more relevant risk will be the ability to re-balance the Δ in order to hedge in the best way the option's exposure. The effectiveness of this re-balancing is strongly affected by the realized volatility of the underlying asset's price. That's why we have to shift our attention on the volatility risk.

First we have to stress the differences between the implied volatility and the realized volatility and the link between these two in affecting the Profit and Loss. In order to do that, we assume that we have a long position in a call option, that is:

$$C = \mathcal{O}(S, t, T, K, \sigma_K, r^d, r^f, 1)$$

and that the book is revaluated at a constant implied volatility σ_K. When the book is Delta-hedged and the Δ constantly re-balanced so to have no exposure to the underlying asset's price movements, which is the main source of risk we have to consider? Or, alternatively said, where the P&L of the book comes from? Those are the questions we try to answer in this section.

Assume that in the real world, and under the true probability measure $\mathbb{P}$, the underlying asset's price evolves according to the following SDE:

$$dS = \mu S dt + \sigma_t S dZ$$

where σ_t is the realized volatility of the process. This volatility may be simply time dependent (that is: deterministically dependent from time), or it can be stochastic. We do not know and we do not want to model the true nature of the process commanding the evolution of the price S. This disregard is translated in a sort of blindly-trusting use of the B&S model at a constant implied volatility σ_K. The implied volatility is just a parameter to put into the formula to get a price, but it also impact on the Delta-hedging strategy, since also the Δ is a function of this parameter. So it seems sensible to think that in some way the performance of the replicating strategy is determined by the implied volatility and its link with the realized volatility. This intuition is confirmed by the following analysis.

We have in our portfolio long position in a call option, and a Δ amount of the underlying to neutralize the exposure to the underlying asset's movements. Over a small period dt the total P&L is given by the actual variations of the price of the option, the actual movement of the underlying asset's price, and the cost of the portfolio financing, f:

$$P\&L = d\Pi + f = dC - \Delta dS + f \tag{4}$$

The actual variation of the value of the option is easily obtained by Ito's Lemma:

$$dC = (\Theta + \frac{1}{2}\sigma_t^2 S^2 \Gamma + \mu\Delta)dt + \Delta\sigma_t S dZ \tag{5}$$

The variation of the underlying asset is trivially dS, whereas the cost for financing the positions is:

$$\begin{aligned} f &= (-r^d(t)C + r^d(t)\Delta S - r^f(t)\Delta S)dt \\ &= (-r^d(t)\Delta S + Df^d r^d(t) K\Phi(d_2) + r^d(t)\Delta S - r^f(t)\Delta S)dt \end{aligned}$$

that is, we pay the interest on the money we borrow to pay the premium C, and we earn or pay the "carry" (depending on the differential between the rates of the two currencies) on the amount Δ in the the underlying asset. In the formula above we just re-write the premium amount C in terms of the B&S formula.

We re-write the formula in section 1 for the Θ of an option as follow (using the definition of Γ defined above):

$$\Theta dt = (-\frac{1}{2}\sigma_K^2 S^2\Gamma + Df^e(t,T)r^f(t)S\Phi(d_1) - Df^d(t,T)r^d(t)K\Phi(d_2))dt$$

we get

$$\Theta dt + f = -\frac{1}{2}\sigma_K^2 S^2 \Gamma dt \tag{6}$$

Substituting (5) and (6) in (4) we obtain

$$P\&L = \frac{1}{2}S^2\Gamma[\sigma_t^2 - \sigma_K^2]dt \tag{7}$$

This is the Profit and Loss resulting form the a Delta-hedging strategy over a small period dt. We make a profit if the realized volatility σ_t higher than the implied volatility σ_K, and the magnitude of this profit is directly linked to the level of the Γ (which is always positive in the case of a plain vanilla call option). If we integrate (7) over the entire option's life, we obtain the total P&L resulting from running a Δ-hedged book at constant implied volatility:

$$P\&L = \int_0^t \frac{1}{2}S_t^2\Gamma(S_t, \sigma_K, t)[\sigma_t^2 - \sigma_K^2]dt \tag{8}$$

Formula (8) is very useful to get some insights about the risks of running a Delta-hedged book. As a very general statement we can say that if we buy an option and hedge it at an implied volatility σ_K lower than the realized volatility σ_t, we make a profit over the entire option's life. But this is not always true, since if the realized volatility is very stochastic and it is higher than the implied volatility in periods when the option has a high Γ, whereas it is lower in periods when the option has low Γ, then the total P&L of the Δ-hedging strategy may turn out to be negative. So, the P&L of the Δ-hedging strategy is highly dependent on the path and on the realized volatility.

We summarize the results of this section in the following:

Facts 3.1.
(1) continuous Δ-hedging of a single option revalued at a constant volatility generates a P&L directly proportional to the Γ of the option;

(2) in general the P&L of a long position in the option, continuously re-hedged, is positive if the realized volatility is, on average during the option's life, higher than the constant implied volatility; it is negative in the opposite case;
(3) the previous statement is not always true since the total P&L is dependent on the path followed by the underlying: if periods of low realized volatility are experienced when the Γ is high, whereas periods of high realized volatility are experienced when the Γ is negligible, then the total P&L is negative, though the realized volatility can be higher then implied volatility for periods longer then those when it is lower.

4. Volatility Risk: Hedging with B&S and Floating Implied Volatility

The analysis of the previous section is very useful to understand the relationship between realized and implied volatility, and how that impact on the P&L of a continuously Delta-hedged portfolio. Nevertheless it cannot be used to estimate the risk in the real world; in fact, everyday the book is marked to the market, so to have a revaluation as near as possible to the true current value of the assets and other derivatives. That means that the book is revaluated at current market conditions regarding the price of the underlying asset and the implied volatility (we drop for the moment the fact that also the interest rates are updated to the current level). What we would like to explore now is the impact on the Delta hedging performance when the implied volatility is floating and continuously updated to the market levels.

Let's start with the following assumptions: Under the real probability measure $\mathbb{P}$, the underlying assets' price evolves according to the following SDE:

$$dS = \mu S dt + \sigma_t dZ_1 \tag{9}$$

The implied volatility σ_K, at which the option is revaluated at any time, is now no more a fixed parameter but is set equal to its the market level; we can then consider the impled volatility a new stochastic factor affecting the option price, and model its evolution, under the real probability measure $\mathbb{P}$, according to the following SDE:

$$d\sigma_K = \alpha dt + \nu_t dZ_2 \tag{10}$$

where dZ_1 and dZ_2 are two correlated Brownian motion.

The two processes above have also an evolution under the equivalent martingale measure $\mathbb{Q}$ provided by the following SDE's:

$$dS = (r^d - r^f) S dt + \sigma_t dW_1 \tag{11}$$

$$d\hat{\sigma}_K = \hat{\alpha} dt + \nu_t dW_2 \tag{12}$$

where dW_1 and dW_2 are again two correlated Brownian motion with correlation parameter ρ.

$$\begin{aligned} dC = \Bigg(& \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma_t^2 S^2 + \frac{\partial C}{\partial S}\mu S \\ & + \frac{\partial C}{\partial \sigma_K}\alpha + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma_K^2}\nu_t^2 + \frac{\partial^2 C}{\partial \sigma_K \partial S}\rho\sigma_t S \nu_t \Bigg) dt \\ & + \frac{\partial C}{\partial S}\sigma_t S dZ_1 + \frac{\partial C}{\partial \sigma_K}\nu_t dZ_2 \end{aligned}$$

and re-writing in terms of the notation we defined in the Introduction

$$\begin{aligned} dC = \big(& \Theta + \frac{1}{2}\Gamma\sigma_t^2 S^2 + \Delta\mu S \\ & + \mathcal{V}\alpha + \frac{1}{2}\mathcal{W}\nu_t^2 + \mathcal{X}\rho\sigma_t S\nu_t \big) dt \\ & + \Delta\sigma_t S dZ_1 + \mathcal{V}\nu_t dZ_2 \end{aligned} \tag{13}$$

Under equivalent martingale measure $\mathbb{Q}$ it can be shown that the dynamics of the call option is described by the SDE

$$\begin{aligned} d\hat{C} &= \Theta + \frac{1}{2}\Gamma\sigma_t^2 S^2 + \Delta(r^d - r^f)S \\ &\quad + \mathcal{V}\hat{\alpha} + \frac{1}{2}\mathcal{W}\nu_t^2 + \mathcal{X}\rho\sigma_t S\nu_t \\ &= r^d\hat{C} \end{aligned} \tag{14}$$

Let's build a portfolio made up by a call option and a quantityΔ of the underlying; it's P&L over a small period dt is:

$$d\Pi = dC - \Delta dS + f \tag{15}$$

where f is the cost born to finance the position: it can be explicitly defined as :

$$f = (-r^d(t)C + r^d(t)\Delta S - r^f(t)\Delta S)dt \tag{16}$$

Substituting in (15) equations (9),(13) and (16), we get

$$d\Pi = \big(\Theta + \frac{1}{2}\Gamma\sigma_t^2 S^2 + \Delta(r^d - r^f)S + \mathcal{V}\alpha + \frac{1}{2}\mathcal{W}\nu_t^2 + \mathcal{X}\rho\sigma_t S\nu_t\big)dt + \mathcal{V}\nu_t dZ_2 \tag{17}$$

adding and subtracting $\mathcal{V}\hat{\alpha}$ and by means of and (10)and (14) we have

$$d\Pi = \mathcal{V}\nu_t dZ_2 + (\mathcal{V}\alpha - \mathcal{V}\hat{\alpha})dt = \mathcal{V}(d\sigma_k - \hat{\alpha}dt)$$

integrating over the option's life:

$$P\&L = \int_0^t d\Pi = \int_0^t \mathcal{V}(d\sigma_k - \hat{\alpha}dt) \tag{18}$$

We can infer form equation (18) that the P&L arising from a continuously Δ-hedged option, revalued at each time by the market prevailing implied volatility, is proportional to the $\mathcal{V}$ and equal to the difference between the actual variation in the implied volatility ($d\sigma_k$) and the expected risk-neutral variation ($\hat{\alpha}dt$, i.e.: the drift of the risk-neutral process). We summarize the results of this section in the following:

Facts 4.1.
(1) continuous Δ-hedging of a single option revalued at a running implied volatility generates a P&L proportional to the $\mathcal{V}$ of the option;
(2) in general the P&L of a long position in the option, continuously re-hedged, is positive if the realized volatility is, on average during the option's life, higher than the constant implied volatility; it is negative in the opposite case;
(3) the previous statement is not always true since the total P&L is dependent on the path followed by the underlying: if periods of low realized volatility are experienced when the Γ is high, whereas periods of high realized volatility are experienced when the Γ is negligible, then the total P&L is negative, though the realized volatility can be higher then implied volatility for periods longer then those when it is lower.

5. Volatility Risk: Exposures to the Volatility Matrix

In the real world the entire volatility surface is stochastic; this does not mean that at any time each implied volatility, corresponding at a given strike and maturity, moves in an erratic way completely unrelated with all the other volatilities.

As a stylized fact, we can identify, for a given expiry, three kind of movements of the smile. Let's start with a flat smile at 10% implied volatility level, depicted in figure 3 as a continuous line. The first supposable movement is a shift upward or downward of the whole smile; for example, if the smile moves to 11%, the resulting new flat smile is drawn as a dashed line in the figure. The second kind of movement is a change in the curvature of the smile; it is a symmetric movement. In figure 4 the 10% flat smile is depicted a continuous line, whereas a dashed line draws the new smile with a positive curvature.

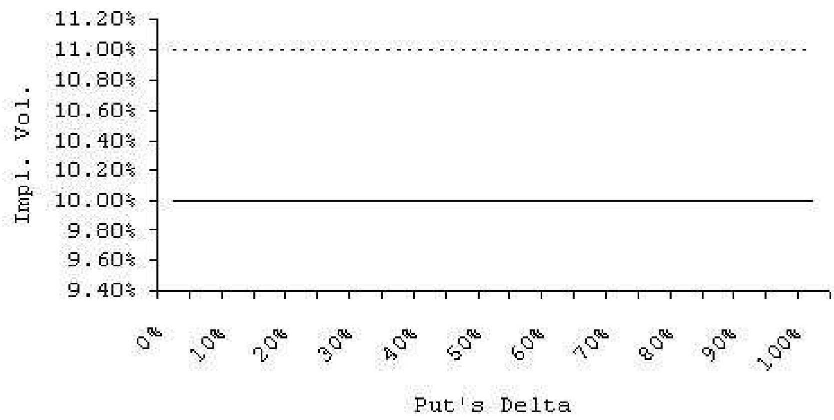

Figure 3. Flat Volatility Smile. Continuous line: 10% implied volatility; dashed line: 11% implied volatility

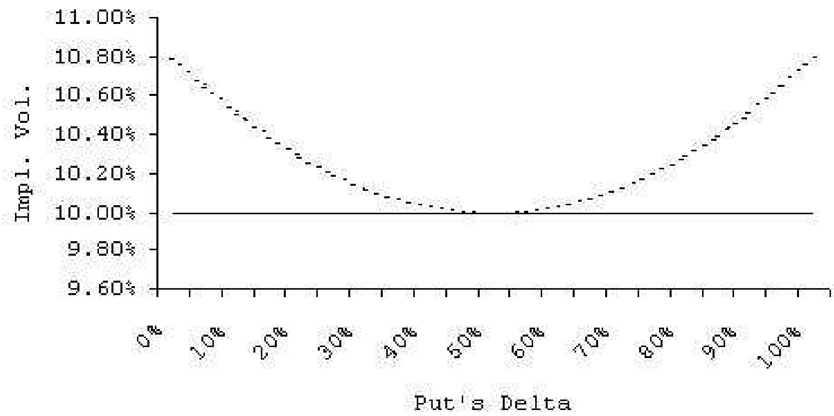

Figure 4. Convex Volatility Smile. Continuous line: 10% flat smile; dashed line: change in the curvature

The third kind of movement is illustrated in figure 5 and is a change in the slope of the volatility smile: the continuous line is the usual flat smile at 10% level and the dashed line is the new smile resulting from a positive increase of the slope. The slope can be also negative; for example in figure 6 the dashed line draws a volatility smile with a a negative change of the slope.

The volatility smile for a given expiry moves according to the three basic movements we have described above. Although it is very useful to disentangle amongst them, one should never forget that in the real world the volatility smile is just the combined result of the three. In figure 7 the dashed line draws a volatility smile produced by a composite change summing up the three basic ones described above. This is a very realistic smile.

From the very quick overview of the basic movements of the smile we may infer that we need a model which is able to cope with the features of the option markets. In what follows we introduce a possible model capable to accomplish this task: it has been designed for FX option markets, but it can be easily adapted to other markets. The model can be considered the simplest extension of the B&S model, and it retains many of the basic characteristics of the latter.

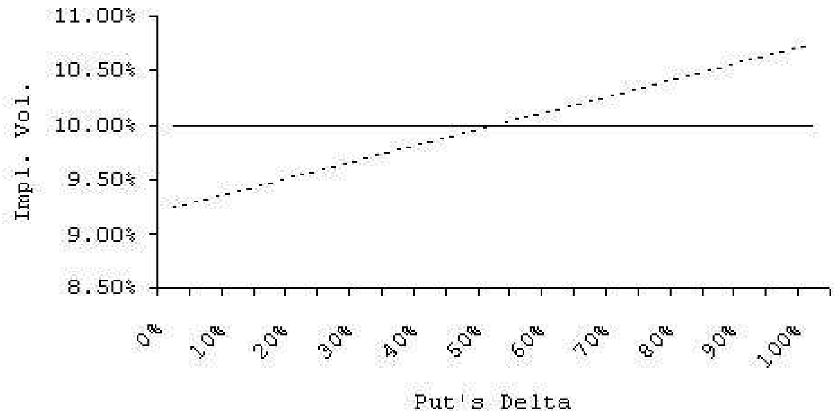

FIGURE 5. Positive Sloped Volatility Smile. Continuous line: 10% flat smile; dashed line: positive increase of the slope

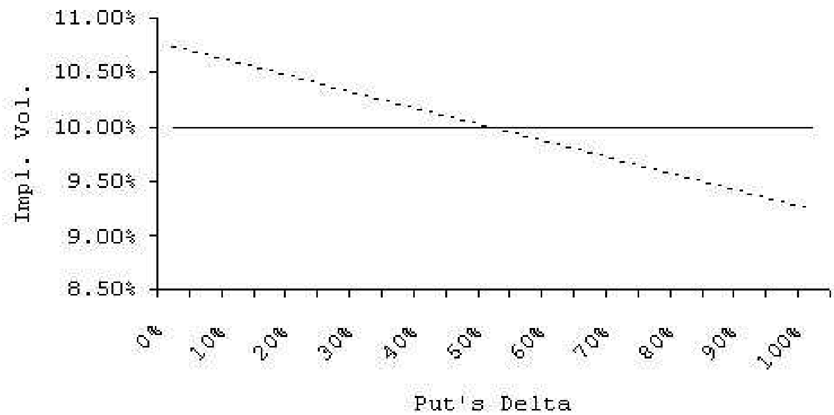

FIGURE 6. Negative Sloped Volatility Smile. Continuous line: 10% flat smile; dashed line: negative increase of the slope

5.1. **The Uncertain Volatility Model.** We assume that the exchange rate dynamics evolves according to the uncertain volatility model with uncertain interest rates (UVUR) proposed by Brigo, Mercurio and Rapisarda (2004). In this model, the exchange rate under the risk neutral measure $\mathbb{Q}$ follows

$$dS(t) = \begin{cases} S(t)[(r^d(t) - r^f(t))\,dt + \sigma_0\,dW(t)] & t \in [0, \varepsilon] \\ S(t)[(\rho^d(t) - \rho^f(t))\,dt + \sigma(t)\,dW(t)] & t > \varepsilon \end{cases} \tag{19}$$

where $r^d(t)$ and $r^f(t)$ are, respectively, the domestic and foreign instantaneous forward rates, W is a standard Brownian motion, and (ρ^d, ρ^f, σ) is a random triplet that is independent of W and

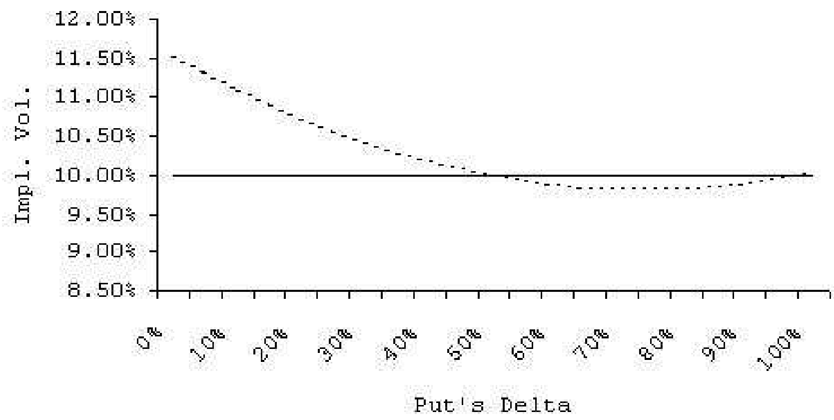

FIGURE 7. Convex & Negative Sloped Volatility smile. Continuous line: 10% flat smile; dashed line: smile resulting from a composite move

takes values in the set of N (given) triplets of deterministic functions:

$$t \mapsto (\rho^d(t), \rho^f(t), \sigma(t)) = \begin{cases} t \mapsto (r_1^d(t), r_1^f(t), \sigma_1(t)) & \text{with probability } \lambda_1 \\ t \mapsto (r_2^d(t), r_2^f(t), \sigma_2(t)) & \text{with probability } \lambda_2 \\ \quad\vdots & \quad\vdots \\ t \mapsto (r_N^d(t), r_N^f(t), \sigma_N(t)) & \text{with probability } \lambda_N \end{cases}$$

where the λ_i are strictly positive and add up to one. The random value of (ρ^d, ρ^f, σ) is drawn at time $t = \varepsilon$.

The intuition behind the UVUR model is as follows. The exchange rate process is just a BS geometric Brownian motion where the asset volatility and the (domestic and foreign) risk free rates are unknown, and one assumes different (joint) scenarios for them.

The volatility uncertainty applies to an infinitesimal initial time interval with length ε, at the end of which the future values of volatility and rates are drawn. Therefore, S evolves, for an infinitesimal time, as a geometric Brownian motion with constant volatility σ_0, and then as a geometric Brownian motion with the deterministic drift rate $r_i^d(t) - r_i^f(t)$ and deterministic volatility $\sigma_i(t)$ drawn at time ε.

In this model, both interest rates and volatility are stochastic in the simplest possible manner. Uncertainty in the volatility is able to accommodate for implied volatility smiles (σ_{RR} close to zero), whereas uncertainty in interest rates can capture skew effects (σ_{RR} far from zero).

Setting $\mu_i(t) := r_i^d(t) - r_i^f(t)$ for $t > \varepsilon$, $\mu_i(t) := r^d(t) - r^f(t)$ and $\sigma_i(t) = \sigma_0$ for $t \in [0, \varepsilon]$ and each i, and

$$M_i(t) := \int_0^t \mu_i(s)\, ds, \quad V_i(t) := \sqrt{\int_0^t \sigma_i^2(s)\, ds}$$

it is easy to show that the density of S at time $t > \varepsilon$ is a mixture of lognormal densities, that is:

$$p_t(y) = \sum_{i=1}^{N} \lambda_i \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y}{S_0} - M_i(t) + \tfrac{1}{2}V_i^2(t) \right]^2 \right\}. \tag{20}$$

As a result, European option prices are mixtures of BS prices as well. For instance the arbitrage-free price of a European call with strike K and maturity T is

(21)
$$Df^d(0,T)\sum_{i=1}^{N}\lambda_i\left[S_0 e^{M_i(T)}\Phi\left(\frac{\ln\frac{S_0}{K}+M_i(T)+\frac{1}{2}V_i^2(T)}{V_i(T)}\right)-K\Phi\left(\frac{\ln\frac{S_0}{K}+M_i(T)-\frac{1}{2}V_i^2(T)}{V_i(T)}\right)\right],$$

where Φ denotes as before the standard normal distribution function. Further details can be found in Brigo, Mercurio and Rapisarda (2004).

The analytical tractability at the initial time is extended to all those derivatives which can be explicitly priced under the BS paradigm. In fact, the expectations of functionals of the process (19) can be calculated by conditioning on the possible values of (ρ^d, ρ^f, σ), thus taking expectations of functionals of a geometric Brownian motion. Denoting by E the expectation under the risk-neutral measure, any smooth payoff G_T at time T has a no-arbitrage price at time $t = 0$ given by

$$G_0 = Df^d(0,T)\sum_{i=1}^{N}\lambda_i E\Big\{G_T\big|(\rho^d = r_i^d, \rho^f = r_i^f, \sigma = \sigma_i)\Big\} = \sum_{i=1}^{N}\lambda_i G_0^{\mathrm{BS}}(r_i^d, r_i^f, \sigma_i) \tag{22}$$

where $G_0^{\mathrm{BS}}(r_i^d, r_i^f, \sigma_i)$ denotes the derivative price under the BS model when the risk free rates are r_i^d and r_i^f and the asset (time-dependent) volatility is σ_i. This model has many desirable characteristics:

- explicit dynamics;
- explicit marginal density at every time (mixture of lognormals with different means and standard deviations);
- potentially perfect fitting to any (smile-shaped or skew-shaped) implied volatility curves or surfaces.
- explicit option prices (mixtures of BS prices) and, more generally, explicit formulas for European-style derivatives at the initial time;
- explicit transitions densities, and hence future option prices;
- explicit (approximated) prices for barrier options and other exotics[1];

We use the UVUR model to consistently estimate the VAR of a complex options' book, containing plain vanilla and exotic options.

5.2. **Calibration of the UVUR Model to the Volatility Smile.** The UVUR model can be easily calibrated to the market volatility surface by the usual techniques, by minimizing the sum of squared percentage differences between model and market volatilities of the 25Δ puts, ATM puts and 25Δ calls, while respecting the following no-arbitrage constraint:

$$\begin{aligned}\sum_{i=1}^{N}\lambda_i e^{-\int_0^t r_i^d(u)\,du} &= Df^d(0,t)\\ \sum_{i=1}^{N}\lambda_i e^{-\int_0^t r_i^f(u)\,du} &= Df^f(0,t)\end{aligned} \tag{23}$$

that is, the current term structure of the interest rates is perfectly matched in the calibration procedure. We provide below an example of calibration to real market FX data, as of 29 May 2005, when the spot exchange rate was 1.2800.

In Table 1 we report the volatility market quotes of EUR/USD for the ATM, and the 25% Δ EUR Call and Putσ_{RR} and σ_{VWB} for the relevant maturities from the overnight (O/N) to two years (2Y), while in Table 2 we report the corresponding domestic and foreign discount factors.

The implied volatility surface that is constructed from the basic volatility quotes is shown in Table 3 and in Figure 8, where for more clearness we plot the implied volatility in terms of put Deltas ranging from 5% to 95% and for the same maturities as in Table 1. A procedure on how

[1] As an expample, a closed-form formula for the price of an up and out call under the UVUR model is reported in Appendix A of Bisesti, Castagna, Mercurio (2002). A complete collection of barrier options formulae under the UVUR model is in Rapisarda (2004)

	$25\Delta p$	ATM	$25\Delta p$
O/N	12.54%	12.00%	11.74%
1W	8.84%	8.50%	8.44%
2W	8.36%	8.10%	8.12%
1M	8.48%	8.27%	8.33%
2M	8.57%	8.40%	8.52%
3M	8.66%	8.55%	8.74%
6M	8.93%	8.85%	9.08%
9M	9.02%	8.95%	9.20%
1Y	9.15%	9.10%	9.36%
2Y	9.26%	9.20%	9.52%

Table 1. EUR/USD volatility quotes as of 09 May 2005.

	T (in years)	$P^d(0,T)$	$P^f(0,T)$
O/N	10-May-05	0.99992	0.99994
1W	16-May-05	0.99941	0.99960
2W	23-May-05	0.99882	0.99920
1M	09-Jun-05	0.99736	0.99823
2M	07-Jul-05	0.99488	0.99654
3M	09-Aug-05	0.99185	0.99461
6M	09-Nov-05	0.98295	0.98938
9M	09-Feb-06	0.97361	0.98363
1Y	09-May-06	0.96432	0.97824
2Y	09-May-07	0.92205	0.95315

Table 2. Domestic and foreign discount factors for the relevant maturities.

	$10\Delta p$	$25\Delta p$	$35\Delta p$	ATM	$35\Delta c$	$25\Delta c$	$10\Delta c$
O/N	13.26%	12.54%	12.26%	12.00%	11.82%	11.74%	11.73%
1W	9.40%	8.84%	8.65%	8.50%	8.43%	8.44%	8.62%
2W	8.86%	8.36%	8.21%	8.10%	8.07%	8.12%	8.38%
1M	8.94%	8.48%	8.35%	8.27%	8.27%	8.33%	8.65%
2M	9.01%	8.57%	8.46%	8.40%	8.43%	8.52%	8.92%
3M	9.04%	8.66%	8.57%	8.55%	8.62%	8.74%	9.20%
6M	9.31%	8.93%	8.86%	8.85%	8.94%	9.08%	9.61%
9M	9.38%	9.02%	8.95%	8.95%	9.04%	9.20%	9.74%
1Y	9.51%	9.15%	9.09%	9.10%	9.20%	9.36%	9.93%
2Y	9.70%	9.26%	9.18%	9.20%	9.32%	9.52%	10.23%

Table 3. EUR/USD volatility quotes as of 09 May 2005.

to consistently build a volatility matrix starting from the three main volatilities available in the market is explained in Castagna and Mercurio (2004). In Table 4 we show the calibration's errors in absolute terms: the model perfectly fits the three main volatilities for each maturity and performs quite well for almost every level of Delta. The perfect calibration to the basic volatility quotes is essential for a breakdown of the volatility exposure along the strike and maturity dimensions. This is extremely helpful to traders and risk managers, since it allows them to understand where their volatility risk is concentrated: this is a crucial point either for hedging or VAR-calculation purposes. The possibility of such a volatility risk breakdown is a clear advantage of the UVUR model.

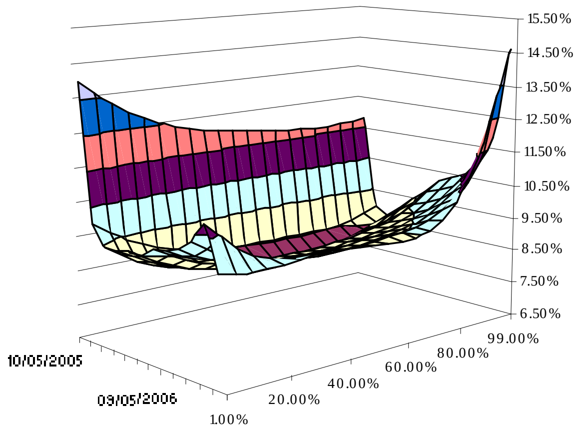

FIGURE 8. EUR/USD implied volatilities (in percentage points) as of 19 May 2005.

	$10\Delta p$	$25\Delta p$	$35\Delta p$	ATM	$35\Delta c$	$25\Delta c$	$10\Delta c$
O/N	-0.05%	0.00%	0.01%	0.00%	0.00%	0.00%	0.00%
1W	-0.04%	0.00%	0.01%	0.00%	0.00%	0.00%	0.00%
2W	-0.04%	0.00%	0.01%	0.00%	0.00%	0.00%	-0.01%
1M	-0.03%	0.00%	0.01%	0.00%	0.00%	0.00%	-0.02%
2M	-0.03%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.03%
3M	-0.02%	0.00%	0.00%	0.00%	0.01%	0.00%	-0.03%
6M	-0.02%	0.00%	0.00%	0.00%	0.01%	0.00%	-0.04%
9M	-0.02%	0.00%	0.00%	0.00%	0.01%	0.00%	-0.04%
1Y	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.04%
2Y	-0.02%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.07%

TABLE 4. Absolute differences (in percentage points) between model and market implied volatilities.

5.3. **Exposures to the Volatility Smile.** After having calibrated the UVUR to the market, it is possible to estimate the exposure of a portfolio to the volatility matrix. The sensitivity to a given implied volatility is readily obtained by applying the following procedure:

- One shifts such a volatility by a fixed amount $\Delta\sigma$;
- then one fits the model to the tilted surface

and calculate the price of the exotic, Π_{NEW}, corresponding to the newly calibrated parameters.

Denoting by Π_{INI} the initial value of the portfolio, its sensitivity to the given implied volatility is thus calculated as:

$$\frac{\Pi_{NEW} - \Pi_{INI}}{\Delta\sigma}$$

For a better sensitivity we can also calculate the portfolio's value under a shift of $-\Delta\sigma$, and then average the two sensitivities.

In practice, it can be more meaningful to hedge the typical movements of the market implied volatility curves, which have been described in the first part of this section. To this end, we start from the three basic data for each maturity (the ATM and the two 25Δ call and put volatilities), and calculate the portfolio's sensitivities to:

- a parallel shift of the three volatilities (at-the-money shift);
- a change in the difference between the two 25Δ wings (risk-reversal shift);

- an increase of the two wings with fixed ATM volatility (butterfly shift).

In this way we should be able to capture the effect of a parallel, a twist and a convexity movements of the implied volatility surface. Once these sensitivities are calculated, it is straightforward use them to hedge or to estimate the VAR. This is exactly done in the following example.

We consider a portfolio consisting in:
(1) Long 10 millions EUR Call expiring in 3 months, struck at 1.35;
(2) Short 20 millions EUR Put expiring in 6m struck at 1.2500;
(3) Long 30 millions EUR Call expiring in 1 year, struck at 1.2800 and a (american) knock out barrier at 1.3600.

We show in Table 5 the relevant sensitivities of the portfolio to the three basic shifts of the volatility matrix mentioned above: the amount for each of them we considered is shown in Table 6.

	Delta	Gamma	Port.Value	Parallel	Rotation	Convexity
O/N	0	0	0	0	0	0
1W	0	0	0	0	0	0
2W	0	0	0	0	0	0
1M	0	0	0	0	0	0
2M	0	0	0	0	0	0
3M	-8,751,789	416,916	545,924	1,516	1,556	1,277
6M	5,834,263	-1,149,826	-251,913	-4,877	3,747	-1,421
9M	0	0	0	0	0	0
1Y	-526,595	-281,308	72,807	-2,012	-521	5,834
2Y	0	0	0	0	0	0
Total	-3,444,121	-1,014,218	366,818	-5,373	4,782	5,691

Table 5. Sensitivities of the portfolio (in EUR).

	Parallelo	Rotazione	Convessit
O/N	0.25%	0.50%	0.02%
1W	0.15%	0.30%	0.01%
2W	0.15%	0.20%	0.05%
1M	0.15%	0.20%	0.05%
2M	0.15%	0.20%	0.05%
3M	0.15%	0.20%	0.05%
6M	0.10%	0.20%	0.05%
9M	0.10%	0.20%	0.05%
1Y	0.10%	0.20%	0.05%
2Y	0.10%	0.20%	0.05%

Table 6. Shifts of the volatility matrix.

Also in this case we do not analyze how to determine the amount of the shifts of the three main volatilities: once again, a wide variety of econometric techniques have been described to forecast the expected changes of economic and financial variables. We refer to them for what concerns this issue.

6. A Combined Underlying Asset's Price-Volatility VAR

After the separated analysis of the two main risks born by an options' portfolio (the underlying and the volatility risk), we have to devise a method to combine them.

We propose here below one possible solution, which is not necessarily the most sophisticated nor the best one:

- We define for the chosen period the expected maximum change of the underlying asset's price at a given confidence level, and the maximum changes of the three main volatilities;
- we use the same procedure described in Section 2 to calculate the value of the portfolio at different asset's price levels, by means of the volatility matrix prevailing in the market, obtaining the N possible $\delta\Pi_i^S$;
- additionally at each asset's price level we calculate also the sensitivity of the portfolio to the forecasted maximum movement of the three main volatility matrix shifts, obtaining $\delta\Pi_i^A, \delta\Pi_i^R, \delta\Pi_i^B$, corresponding, respectively, to the at-the-money, risk-reversal and butterfly shift. Let's define the total volatility exposure at the level of the asset's price S_i as $\delta\Pi_i^V = |\delta\Pi_i^A| + |\delta\Pi_i^R| + |\delta\Pi_i^B|$, that is the sum in absolute terms of the three exposures;
- Estimate the VAR by picking the maximum negative variation of the portfolio due both to the underlying asset's and to the volatility surface's movements: $VAR = \min_{i=1,\dots,N}(\delta\Pi_i^S - \delta\Pi_i^V)$. That means that for each level of the price we subtract to the value of the portfolio the variations due the possible adverse movements of the volatility matrix.

The procedure is quite conservative since it does not make any assumption regarding the correlation between the asset's price and the deformation of the volatility matrix. For example, one may assume that, with price going up, the matrix would experience a twist with a more pronounced positive slope: this makes quite sense and it is a good description of the common market's behavior. But, on the other hand, one may also argue that the task of a VAR estimation is just to calculate the risk when prices and financial variables move in a total unexpected and unusual way. So, it may be considered a conservative VAR estimation the brute sum of the worst results given the predicted maximum excursions of the relevant variables affecting the value of our portfolio.

The procedure is quite conservative since it does not make any assumption regarding the correlation between the asset's price and the deformation of the volatility matrix. For example, one may assume that, with price going up, the matrix would experience a twist with a more pronounced positive slope: this makes quite sense and it is a good description of the common market's behavior. But, on the other hand, one may also argue that the task of a VAR estimation is just to calculate the risk when prices and financial variables move in a total unexpected and unusual way. So, it may be considered a conservative VAR estimation the brute sum of the worst results given the predicted maximum excursions of the relevant variables affecting the value of our portfolio. We show a simplified example of the procedure we proposed. In Table 7 we describe the variations in the value if the portfolio for different levels of the underlying asset price: we start from 1.2800, the same level prevailing in the market when we performed the calibration of the UVUR model. The chosen levels, departing from the starting price, are equal to ±1%,±2% and ±4%. Trivially the variation is nil at 1.2800, but nonetheless a VAR can be calculated even with stale asset's price since we have an exposure to the volatility matrix. Hence the total VAR at 1.2800 is $VAR = 0 + |-5,373| + |4,782| + |5,691| = 15,846$; we can go on and calculate the VAR for all the levels of the price, yielding the results presented in Table 8. The total VAR of the portfolio is then $-59,605$ EUR.

6.1. **Some *Caveat* in Estimating the VAR.** The combined approach described above to estimate the VAR is an abstract concept of the real world, which does not take into account many features of the actual running of an options' portfolio. This is true in general for any approach, and what follows should be thoroughly considered in any designing any procedure to calculate the VAR.

(1) In estimating the VAR, one assumes that, for the chosen period, the portfolio retains the same sensitivity, to the asset's price and to the volatility matrix, for the entire period: this can be reasonable only if the chosen time horizon is very short. If we want to calculate the VAR for relatively long periods we should consider the change of the sensitivities of the portfolio due to the passing of time. We could split the time period in many sub-periods and revaluate the portfolio and calculate the sensitivities at each of them: after that we apply the procedure described above for each sub-period, and estimate the total VAR of the period as the maximum VAR obtained.

(2) Another implicit assumption in estimating the VAR is that no new positions are taken during the chosen period; once again, this could be a negligible assumption only for very short periods,

but in some cases it is always a very unrealistic assumption, since at least Δ-hedging is a very frequent activity almost in every moment changing the structure of the portfolio. One may argue that the very essence of the VAR is that the risk is referred to a given portfolio, whose components do not change over time, and we may also yield to that: but that is the reason why we claimed above that the VAR is an abstract concept. Anyway, besides all the philosophical considerations, when calculating the VAR one should include at least also the contingent orders a trader left to the market. This order may be left for re-balancing the Δ exposure of the portfolio when some levels of the asset's price are reached. Moreover, FX options' book may include barrier options, and when the barrier is breached, a substantial change in Δ exposure usually arises: so, huge contingent orders are often left to the (interbank FX) market, to buy or sell the underlying currency when the level corresponding to a barrier is breached. When calculating the VAR for some asset's price levels some barrier options have to be revaluated as knocked (getting as likely result a dramatic change in the Δ),the related contingent orders should be included in estimation procedure (if they have been left to the market, clearly).
(3) We did not include in the VAR procedure the change of the value of the portfolio due to the passing of time, that is the Θ. The reason is that the Θ is not a risk: it is a cost. When the chosen time horizon is short, the Θ should can be simply added to the total VAR. When we want to estimate the VAR for longer period, if we accept the suggestion described in point one, we take implicitly into account also the Θ, since we revaluate the portfolio at different times in the future.
(4) Even if we proposed a VAR estimation procedure based on variations of the assets' price and of the volatility matrix predicted with a given confidence level, we prefer always to put the portfolio under a stress analysis, that is: verify which is the performance of the portfolio for extreme and *ex ante* unrealistic movements of the price and of the volatility matrix. For example, in the exercise presented above, one may calculate what happens to the portfolio if the price goes to 0 or to 5; or the ATM volatility collapse next to 0 or explode to 100%. This may seem a futile calculation, but in the end our personal opinion is that this is the only *true* VAR of a portfolio.

7. Other Risks

In the previous analysis we did not include in the VAR estimation procedure some risks, some of which can be considered negligible, and some much more relevant.

First, we did not include the risk arising from the exposure to the domestic and foreign interest rates: although this exposure has usually a very limited impact on the value of the portfolio, in some situations it can be sensible to include also the interest rate's risk into the VAR. The extension of the procedure described above is quite easy.

Second, and probably more important, a basic assumption in the VAR estimation is that the model by which we revaluate the portfolio in the different scenarios is a good predictor. This cannot be taken as a realistic assumption, and the model risk becomes a new risk we must add into the VAR. Some new research has been put forward in analyzing this issue by Cont (2005), although we are just at the beginning and it will likely be one of the most studied aspects of the financial institutions.

8. Conclusions

In this document we described the risks that mainly impact the value of a portfolio of FX options: the exposure to the underlying asset's price and to the volatility. We suggested some practical solutions to the VAR estimation related to these risks; we also shed some light on the relationships between the asset's price movements and (both realized and implied) volatility, in determining the P&L of a frequently re-balanced portfolio.

We do not claim it is an exhaustive research and many aspect are deferred to future research.

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	1.2800				1.2928			
	Price Chge	Parallel	Rotation	Convexity	Price Chge	Parallel	Rotation	Convexity
O/N	0	0	0	0	0	0	0	0
1W	0	0	0	0	0	0	0	0
2W	0	0	0	0	0	0	0	0
1M	0	0	0	0	0	0	0	0
2M	0	0	0	0	0	0	0	0
3M	0	1,516	1,556	1,277	-89,779	1,899	1,633	1,085
6M	0	-4,877	3,747	-1,421	54,796	-4,408	4,421	-2,171
9M	0	0	0	0	0	0	0	0
1Y	0	-2,012	-521	5,834	-7,307	-1,923	124	5,742
2Y	0	0	0	0	0	0	0	0
Total	0	-5,373	4,782	5,691	-42,290	-4,433	6,178	4,656

	1.3056				1.3312			
	Price Chge	Parallel	Rotation	Convexity	Price Chge	Parallel	Rotation	Convexity
O/N	0	0	0	0	0	0	0	0
1W	0	0	0	0	0	0	0	0
2W	0	0	0	0	0	0	0	0
1M	0	0	0	0	0	0	0	0
2M	0	0	0	0	0	0	0	0
3M	-94,251	2,290	1,523	768	-102,937	2,884	685	121
6M	56,729	-3,895	4,790	-2,868	60,482	-2,877	4,728	-3,782
9M	0	0	0	0	0	0	0	0
1Y	-7,949	-1,712	619	5,190	-9,197	-995	913	3,020
2Y	0	0	0	0	0	0	0	0
Total	-45,471	-3,317	6,931	3,091	-51,651	-987	6,326	-641

	1.2672				1.2544			
	Price Chge	Parallel	Rotation	Convexity	Price Chge	Parallel	Rotation	Convexity
O/N	0	0	0	0	0	0	0	0
1W	0	0	0	0	0	0	0	0
2W	0	0	0	0	0	0	0	0
1M	0	0	0	0	0	0	0	0
2M	0	0	0	0	0	0	0	0
3M	95,822	1,175	1,351	1,316	102,37	6,548	1,086	1,221
6M	-67,499	-5,259	2,783	-733	-70,758	-3,259	1,583	-230
9M	0	0	0	0	0	0	0	0
1Y	4,632	-1,975	-1,237	5,462	5,422	790	-1,926	4,678
2Y	0	0	0	0	0	0	0	0
Total	32,956	-6,059	2,897	6,044	37,035	4,079	742	5,670

	1.2288			
	Price Chge	Parallel	Rotation	Convexity
O/N	0	0	0	0
1W	0	0	0	0
2W	0	0	0	0
1M	0	0	0	0
2M	0	0	0	0
3M	115,877	479	579	829
6M	-77,480	-5,543	-1,122	-110
9M	0	0	0	0
1Y	7,052	-1,280	-2,933	2,326
2Y	0	0	0	0
Total	45,449	-6,344	-3,475	3,045

Table 7. Variations of the portfolio values for different scenarios of asset's price and volatility shifts (in EUR).

	1.2672	1.2544	1.2288	1.2800	1.2928	1.3056	1.3312
Chg of Port.lio Value	17,956	24,174	32,583	-15,846	-57,556	-58,810	-59,605

TABLE 8. Value at risk of the portfolio at different asset's price levels (in EUR).