Towards a Theory of Internal Valuation and Transfer Pricing of Products in a Bank: Funding, Credit Risk and Economic Capital*

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Abstract

We sketch a framework to theoretically identify the components of the value that a bank should attach to a deal and how to charge them to the relevant departments and/or to the final counterparty (client) by an internal transfer pricing system.

1 Introduction

The pricing of contracts in the financial industry relies on theoretical results in many cases. Suffice to remember that the valuation of derivative contracts hinges on the results of Option Pricing Theory, mainly developed between the 1970's and the 1990's.

In some cases the practice differs from the theory. For example, the inclusion of funding costs in the valuation of loans has been seen in striking contrast with the results proved by academicians such as Modigliani and Miller, awarded with Nobel prizes for their achievements.

In the last few years, the debate has sparked also as far as derivative contracts are concerned: there is not a widespread agreement on whether adjustments such as the Debit Value Adjustment (**DVA**) and Funding Value Adjustments (**FVA**) are justified or not. Some authors have proposed full pricing approaches, including every possible adjustment in the value: we can refer to the most recent works in this area by Brigo, Morini and Pallavicini [1] and Crepy *et al.* [11]. It should be noted that these authors are more interested in finding out a valuation formula encompassing all the components of the pricing as seen by both counterparties, but they do not investigate which components can be effectively replicated so that they represent a recoverable production cost. In few words, these approaches aim at determining a fair tradable price on which both parties

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may agree, by acknowledging each others' risks and costs, but they cannot always be correct and consistent methods to evaluate derivative contracts once they are traded and entered in the bank's books.

The heart of the matter with the revaluation of derivative contracts is in the assumptions the bank makes about its hedging policies: they can be more or less realistic and feasible, so that different evaluations can be obtained, which have to be challenged by the ability to actually implement the assumed hedging strategies. In this work we will try to shed some light on how products (contracts) should be internally evaluated by a bank, on which are the costs that should be transferred to the counterparty, and on the extent to which the theoretical results still applies in practice and under which conditions they do so. To this end, we will refer to a stylised balance sheet in a basic multi-period setting.

Some of the results we will crop up in the present work are derived in Castagna [8], but within a framework that, although similar to the one we will sketch below, was not capable to avoid an aporetic situation that was however acknowledged in the final part of that paper. By modifying and expanding said framework, we will hopefully provide a firmer theoretical ground to some of the claims made elsewhere in Castagna [7], and confirm the other results in Castagna [5].

2 Balance Sheet with a Single Asset

Let us start with the assumption that we are in an economy with interest rates set constant at zero level. An asset $A_1(t)$ has an initial price X_1 at time t, terminal pay-off $A_1(T) = X_1 \times (1 + s_1)$ (for simplicity, we will assume that $T - t = 1$ in what follows). There is a probability $\mathbf{PD}_1$ that the asset's issuer defaults between times t and T, in which case the buyer of the asset will receive in T a stochastic recovery $\mathbf{Rec}_1$ expressed as a percentage of the face value X_1. $\mathbf{Rec}_1$ can take values $\mathbf{Rec}_{1_j}$, for $j = 1, ..., J$, and each possible value can occur with probability p_{1_j}.

$$A_1(t) = \mathbf{E}\left[X_1 \times (1 + s_1)\right] = X_1 \times (1 + s_1)(1 - \mathbf{PD}_1) + X_1 \overline{\mathbf{Rec}}_1 \mathbf{PD}_1 = X_1 \tag{1}$$

where $\overline{\mathbf{Rec}}_1 = \sum_{j=1}^{J} \mathbf{Rec}_{1_j} p_{1_j}$ is the expected recovery in the event of the issuer's bankruptcy.

Assume that an investor (with zero leverage) buys the asset A_1. In perfect markets, the spread s_1 they require from the issuer of the asset A_1 is simply the fair credit spread cs_1 remunerating the credit risk:

$$s_1 = cs_1 = \frac{\overline{\mathbf{Lgd}}_1 \mathbf{PD}_1}{1 - \mathbf{PD}_1} \tag{2}$$

where $\overline{\mathbf{Lgd}}_1 = (1 - \overline{\mathbf{Rec}}_1)$ is the loss given default rate (the complementary to 1 of the recovery rate). The spread s_1 is set at the level that makes the terminal expected value of A_1 equal to the present value, X_1.

Assume now that a bank is established at time t and wants to invest in the asset A_1. The bank uses leverage, which means that it issues a bond to buy the asset. We denote the value of the debt at time t with $D_1(t)$ and the amount of equity posted by shareholders with E. The equity is invested in a risk-free bank account, $B(t) = E$. The amount needed to fund the asset is $D_1(t) = X_1$: this amount is raised by the bank with a bond issuance. The terminal pay-off of the bank's debt is linked to the pay-off of asset

A_1. In fact, the debt D_1 pays at maturity $X_1 \times (1 + f_1)$ if the asset's issuer does not go defaulted, or $X_1\mathbf{Rec}_{1_j} + E$ (for each possible recovery rate) if it does. In a perfect market, with perfectly informed agents, the fair bank's funding spread, requested by the buyers of the bank's bond, is derived from the following equation:

$$\begin{aligned} D_1(t) &= \mathbf{E}\left[X_1 \times (1 + f_1)\right] \\ &= X_1 \times (1 + f_1)(1 - \mathbf{PD}_1) + \sum_{j=1}^{J} \min[X_1\mathbf{Rec}_{1_j} + E; X_1 \times (1 + f_1)]p_{1_j}\mathbf{PD}_1 \\ &= X_1 \end{aligned} \tag{3}$$

The LHS of equation (3) is the expected value of the bank's bond at expiry: if the issuer of the asset A_1 survives, then the bank is able to reimburse the notional of the bond plus the funding spread $X_1 \times (1 + f_1)$. If the assets' issuer goes bust, then the bond will be reimbursed with the funding spread only if this amount is smaller than the recovery value of the asset plus the initial equity deposited in the risk-free account: this is computed for each possible value of $\mathbf{Rec}_{1_j}$ and weighted for the corresponding probability. This expected value must equal the present value of the debt, which is the face amount X_1.

We assume that there is at least one case when $E < X_1\mathbf{Lgd}_{1_j} + X_1 f_1$ (the equity is not able to fully cover the loss given the default of the asset's issuer and the funding cost). The funding spread is the level of f_1 equating the expected value at expiry with the present value:

$$f_1 = \frac{\overline{\mathbf{Lgd}}_1^* \mathbf{PD}_1}{1 - \mathbf{PD}_1} \tag{4}$$

where $\overline{\mathbf{Lgd}}_1^* = 1 - \sum_{j=1}^{J} \min[X_1\mathbf{Rec}_{1_j} + E; X_1 \times (1 + f_1)]p_{1_j}/X_1$.[1] The quantity $\overline{\mathbf{Lgd}}_1^*$ can be also be seen as the expected loss given default on the bank's debt, $\overline{\mathbf{Lgd}}_1^* = \overline{\mathbf{Lgd}}_{D_1}$, and the corresponding recovery is $\overline{\mathbf{Rec}}_{D_1} = \sum_{j=1}^{J} \min[X_1\mathbf{Rec}_{1_j} + E; X_1 \times (1 + f_1)]p_{1_j}/X_1$. They are different from the loss given default and the recovery of the asset A_1 since when the asset's issuer defaults, a fraction of the loss is covered by the equity E, which is also the maximum amount the shareholders of the bank are liable for. Since $\overline{\mathbf{Lgd}}_1 > \overline{\mathbf{Lgd}}_1^* = \overline{\mathbf{Lgd}}_{D_1}$, then $s_1 > f_1$, the bank "enjoys" (if an economic agent may ever enjoy from losing a part of its own capital) from the fact that the equity is contributing to the coverage of the potential future credit losses. If $E > X_1\mathbf{Lgd}_{1_j} + X_1 f_1$ for all possible $\mathbf{Lgd}_{1_j}$ then $\overline{\mathbf{Lgd}}_1^* = \overline{\mathbf{Lgd}}_{D_1} = 0$ and $f_1 = 0$.

Assume now that the bank has a bargaining power in setting the fair rate s_1, so that it is able to set it at a level different from the fair level cs_1 required by a non-leveraged investor. In this way the bank tries to cover costs and losses other than the credit losses. The fair mark-up spread ms_1 the bank has to charge on asset A_1, given the limited liability

[1] It should be noted that equation (4) is formally the solution to equation (3), but it is not a closed-form formula, since the definition of $\overline{\mathbf{Lgd}}_1^*$ includes f_1. That means that a numerical search for f_1 is needed in (3), starting by setting $f_1 = 0$: for practical purposes and for typical values of the involved variables, two steps are sufficient.

of the shareholders, is obtained by the equation:

$$\begin{aligned}\mathbf{VB}(t) &= \mathbf{E}\left[X_1 \times (1+ms_1) + E - X_1 \times (1+f_1)\right] \\ &= [(ms_1 - f_1)X_1 + E](1-\mathbf{PD}_1) + \sum_{j=1}^{J} \max[X_1 \mathbf{Rec}_{1_j} + E - X_1 \times (1+f_1); 0] p_{1_j} \mathbf{PD}_1 \\ &= E\end{aligned} \tag{5}$$

Equation (5) is the net value of the bank at time t, which is equal to the expected value of the future value in T of the bank's total assets (the final asset's pay-off X_1 plus the margin $ms_1 X_1$, plus the equity amount deposited in the risk-free account E), minus the bank's total liabilities (the notional of the debt X_1 plus the funding costs $f_1 X_1$). Since the bank's shareholders invested the initial amount E, the expected net value $\mathbf{VB}(t)$ must equal E to prevent any arbitrage.

Assume $E < X_1 \mathbf{Lgd}_{1_j} + X_1 f_1$ for one or more $\mathbf{Lgd}_{1_j}$'s. Indicating with

$$\overline{R}_1 = \sum_{j=1}^{J} \max[X_1 \mathbf{Rec}_{1_j} + E - X_1 \times (1+f_1); 0] p_{1_j},$$

the mark-up spread, from (5), is:

$$ms_1 = \frac{\frac{E-\overline{R}_1}{X_1}\mathbf{PD}_1}{1-\mathbf{PD}_1} + f_1 = cs_1^* + f_1 \tag{6}$$

This is the sum two components:

- the "adjusted" credit spread $cs_1^* < cs_1$ on the asset A_1, due the loss given default $(E - \overline{R}_1)/X_1$ lower than $\mathbf{Lgd}_1$. The smaller loss given default suffered by the shareholders is generated by the leveraged investment in the asset A_1, and by the limited liability up to the equity amount E. In practice a share of the losses given default is taken by the debt holders, whence the lower credit spread;
- the funding spread f_1 paid by the bank on its debt.

By some manipulations, it is easy to check that in perfect markets where the credit spreads set by investors are fair and given in (2), we have:

$$ms_1 = \frac{\frac{E-\overline{R}_1}{X_1}\mathbf{PD}_1 + \overline{\mathbf{Lgd}}_1^*\mathbf{PD}_1}{1-\mathbf{PD}_1} = \frac{\overline{\mathbf{Lgd}}_1\mathbf{PD}_1}{1-\mathbf{PD}_1} = s_1 \tag{7}$$

So the mark-up spread is just the credit spread of the asset A_1 required by a for a non-leveraged investor.

If $E > X_1 \mathbf{Lgd}_{1_j} + X_1 f_1$ for all possible $\mathbf{Lgd}_{1_j}$ (the equity is large enough to cover the loss given the default of the asset's issuer and the funding costs in each possible case), then $f_1 = 0$ and $\sum_{j=1}^{J} \max[X_1 \mathbf{Rec}_{1_j} + E - X_1 \times (1+f_1); 0] p_{1_j} = X_1 \overline{\mathbf{Rec}}_1 + E - X_1 \times (1+f_1)$. From equation (5) we have that the mark-up spread is $ms_1 = (\overline{\mathbf{Lgd}}_1 \mathbf{PD}_1)/(1-\mathbf{PD}_1)$, which is consistent with the stated results.

Proposition 2.1. *If the bank holds only one asset, the leverage is immaterial in its internal pricing. Differently stated, the bank can price the asset as it were an non-leveraged investor, and the assets' price would depend only on its expected future pay-off.*

The bank's default probability $\mathbf{PD}_1$ is linked to the issuer's probability of default $\mathbf{PD}_1$, and it depends on the amount of equity that can be used to lower the $\mathbf{Lgd}_{1_j}$ in the different cases. In general terms, the bank defaults when its value is below zero: since the shareholders' liability is limited, the bank's value is floored at zero, so that the bank goes bust when $R_{1_j} = \max[X_1\mathbf{Rec}_{1_j} + E - X \times (1 + f_1); 0] = 0$, for one or more j's. We have that:

$$\mathbf{PD}_B = \sum_{j=1}^{J} \mathbf{1}_{\{R_{1_j}=0\}} p_{1_j} \mathbf{PD}_1 \tag{8}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function.

The conclusion is that bank's leverage and funding costs are immaterial in pricing the asset A_1: the requested spread to make fair the contract is the same both for a non-leveraged investor and for the leveraged bank. This result is definitely not new: in a different setting, we reached the same conclusions as the well known works by Modigliani&Miller (M&M) [17] and Merton [16], whose results remain fully valid also in our setting.

When we are not in perfect markets, formula (6) provides the bank with a useful tool to price the asset A_1. In fact, it is possible that the debt holders do not have access to the information set of the bank, so that they do not know which is the correct level of $\mathbf{PD}_1$ and $\mathbf{Rec}_1$ to apply in the pricing. For this reason, the funding spread required on the bank's debt can be different from the fair spread applied in a market where a perfect information is available to all market participant.

Let $f_1^* \neq f_1$ be the funding spread required by the debt holders. The equivalence in equation (7) does not hold anymore (and so does not also the M&M theorem and its extension by Merton). The bank should use equation (6), where f_1 is set equal to the spread requested by the debt holders f_1^*, and the credit spread is the "enhanced" spread cs_1^*.

Formula (6) allows also to consider different risk-premia implicit in the preferences of the shareholders and bond holders: in perfect markets, in the simple framework we are working in, the risk-premia are reflected in the **PD** of the asset's issuer. In the pricing process, if the bond holders use a different **PD** from the one used by shareholders (or by the bank's managers, if they operate mainly in the interest of the latter), then the irrelevance of the leverage does not hold anymore and the correct formula to use. Different **PD**s are the consequence of different premia due to a limited information of the bond holders about the issuer of the asset bought by the bank.[2]

[2]It is quite reasonable to assume that typically banks have a deeper knowledge of their clients, which are ultimately the issuers of most of the assets. On the other hand, when the asset is a publicly traded security, it is more reasonable to assume that the information is evenly distributed amongst market participants. This is the case implicitly assumed in the M&M's and Merton's works.

2.1 Including the Economic Capital

The economic capital is a concept used by financial institutions to indicate the amount of equity capital needed to keep the probability of default (*i.e.*: of ending the business activity) below a given level, according to a model to measure the future uncertainty of the assets' value.[3]

We can embed a premium in the return on equity (upon the risk-free rate) different from the one implicitly originated by the use of a **PD** containing a risk-premium fair to the bank. Let us indicate by π this additional return on equity and with $\mathbf{EC}_1 \leq E$ the amount of economic capital "absorbed" by asset A_1. Equation (5) modifies as follows:

$$\begin{aligned}\mathbf{VB}(t) &= \mathbf{E}\left[X_1 \times (1 + ms_1) + E - X_1 \times (1 + f_1)\right] \\ &= [(ms_1 - f_1)X_1 + E](1 - \mathbf{PD}_1) + \sum_{j=1}^{J} \max[X_1 \mathbf{Rec}_{1_j} + E - X \times (1 + f_1); 0] p_{1_j} \mathbf{PD}_1 \\ &= \mathbf{EC}_1 \pi + E\end{aligned} \tag{9}$$

The margin spread including this additional premium is:

$$ms_1 = \frac{\frac{E - \overline{R}_1}{X_1}\mathbf{PD}_1 + \mathbf{EC}_1/X_1\pi}{1 - \mathbf{PD}_1} + f_1 = cs_1^* + f_1 + cc_1 \tag{10}$$

where $cc_1 = \frac{\mathbf{EC}_1/X_1\pi}{1-\mathbf{PD}_1}$ is the cost of the economic capital associated to asset A_1.

Proposition 2.2. *In internally evaluating an asset held by the bank, the economic capital enters in the bank's internal pricing process through the return π requested by the shareholders.*

2.2 Including Interest Rates Different from Zero

It is relatively easy to introduce non-zero interest rates in the framework sketched above: we limit the analysis to the case of a single, constant interest rate equal to r. In this case equation (1) modifies as follows:

$$A_1(t) = \mathbf{E}\left[\frac{1}{1+r} X_1 \times (1 + r + s_1)\right] = \frac{X_1 \times (1 + r + s_1)(1 - \mathbf{PD}_1) + X_1 \overline{\mathbf{Rec}}_1 \mathbf{PD}_1}{1 + r} = X_1 \tag{11}$$

After same manipulations (see also Castagna and Fede [9], ch. 8), we have a formula for the credit spread s_1 similar to formula (2), where the loss given default is defined as $\overline{\mathbf{Lgd}}_1 = 1 + r - \overline{\mathbf{Rec}}_1$; for typical values of the interest rates, the two $\overline{\mathbf{Lgd}}_1$'s do not differ too much.

[3] The level is often set by national and international regulation, such as the Basel regulation for banks.

Similarly, when rates are different from zero, the bank's debt equation (3) reads:

$$\begin{aligned} D_1(t) =& \mathbf{E}\left[\frac{X_1 \times (1+r+f_1)}{1+r}\right] = \frac{X_1 \times (1+r+f_1)(1-\mathbf{PD}_1)}{1+r} \\ &+ \frac{\sum_{j=1}^{J} \min[X_1 \mathbf{Rec}_{1_j} + E(1+r); X_1 \times (1+r+f_1)] p_{1_j} \mathbf{PD}_1}{1+r} \\ &= X_1 \end{aligned} \tag{12}$$

so that the funding spread has still a formula as in (4), with

$$\overline{\mathbf{Lgd}}_1^* = 1 + r - \sum_{j=1}^{J} \min[X_1 \mathbf{Rec}_{1_j} + E(1+r); X_1 \times (1+f_1)] p_{1_j} / X_1$$

.

Finally, the fair mark-up spread ms_1 the bank has to charge on asset A_1 when rates are different from zero, is given by solving the following equation:

$$\begin{aligned} \mathbf{VB}(t) = & \mathbf{E}\left[\frac{X_1 \times (1+r+ms_1) + E(1+r) - X_1 \times (1+r+f_1)}{1+r}\right] \\ =& \frac{[(ms_1 - f_1)X_1 + E(1+r)](1-\mathbf{PD}_1)}{1+r} \\ &+ \frac{\sum_{j=1}^{J} \max[X_1 \mathbf{Rec}_{1_j} + E(1+r) - X_1 \times (1+f_1); 0] p_{1_j} \mathbf{PD}_1}{1+r} \\ =& E \end{aligned} \tag{13}$$

The formula for ms_1 is similar as (6), if $E(1+r) < X_1 \mathbf{Lgd}_{1_j} + X_1 f_1$ for one more $\mathbf{Lgd}_{1_j}$'s:

$$ms_1 = \frac{\frac{E(1+r) - \overline{R}_1}{X_1} \mathbf{PD}_1}{1 - \mathbf{PD}_1} + f_1 = cs_1^* + f_1 \tag{14}$$

where $\overline{R}_1 = \sum_{j=1}^{J} \max[X_1 \mathbf{Rec}_{1_j} + E(1+r) - X_1 \times (1+f_1); 0] p_{1_j}$.

If $E(1+r) < X_1 \mathbf{Lgd}_{1_j} + X_1 f_1$ in all cases, then $ms_1 = (\overline{\mathbf{Lgd}}_1 \mathbf{PD}_1)/(1 - \mathbf{PD}_1)$ with $\overline{\mathbf{Lgd}}_1 = 1 + r - \overline{\mathbf{Rec}}_1$.

In conclusion, the introduction of interest rates into the analysis does not significantly affect results. Let the total yield of the asset A_1 be i_1: it can be written as the sum of the following components:

$$i_1 = r + ms_1 = r + cs_1^* + f_1 + cc_1 \tag{15}$$

We present now an example to illustrate in practical terms what we have stated in this section.

Example 2.1. *Assume that the bank starts its activity in $t = 0$ with an amount of equity capital $E = 35$, which is deposited in a bank account $B(0) = 35$. The bank wishes to invest in an asset whose price is $A_1(0) = 100$ and it issues debt $D_1(0) = 100$ to buy it. We will suppress the time reference in the following to lighten the notation. A sketched bank's balance sheet is the following:*

Assets	**Liabilities**
$B = 35$ $A_1 = 100$	$D_1 = 100$ ———————— $E = 35$

We assume interest rates are zero and that the issuer of the asset A_1 can default with probability $\mathbf{PD} = 5\%$. Moreover, upon default, the recovery rate for the bank is stochastic: possible outcomes and the associated probabilities are in table 1. Clearly, $\overline{\mathbf{Lgd}}_1 = 1 - \overline{\mathbf{Rec}}_1 = 60\%$

	$\mathbf{Rec}_{1_j}$	p_j
	75%	20%
	35%	70%
	5%	10%
Exp. Recovery $\overline{\mathbf{Rec}}_1$	40%	

Table 1: Possible recovery rates $\mathbf{Rec}_{1_j}$ and associated probabilities p_{1_j} for asset A_1

Firstly, let us determine which is the fair (credit) spread $s_1 = cs_1$ requested by a non-leveraged investor. This is given by equation (2):

$$s_1 = cs_1 = \frac{60\% \times 5\%}{1 - 5\%} = 3.158\%$$

so that at the expiry the asset has a terminal value of $A_1 = 100 \times (1 + 3.158\%)$.

The bank is a leveraged investor, since it issues an amount of debt sufficient to buy the asset. Assuming we are in a market where perfect information is available to all participants, then the creditors of the bank know that it will buy the asset A_1 and consequently they set a credit spread on the debt D_1, which is a funding spread for the bank, by applying equation (4), by means of a two-step iterative procedure (the first step is computed by setting $f_1 = 0$), so that we get:

$$f_1 = 1.406\%$$

We can now compute the fair margin ms_1 that the bank should charge on asset A_1, by means of formula (6). We start with computing the different R_{1_j}s: they are shown in the table below. Since with these quantities we can compute also the bank's default probability, by means of formula (8), this is shown in the table below as well.

R_{1_j}	PD_B
1.719	*0.000%*
-	*3.500%*
-	*0.500%*
1.719	*4.000%*

We have all we need to calculate the "adjusted" credit spread cs_1^*,

$$cs_1^* = \frac{\frac{35-1.79}{100}5\%}{1-5\%} = 1.752\%$$

which plugged in (6)

$$ms_1 = cs_1^* + f_1 = 1.752\% + 1.406\% = 3.158\% = s_1$$

thus confirming (7).

The equity should be used to prevent a default of the bank occurring with a probability higher than a given level. The economic capital is the amount of equity necessary to keep the probability below the targeted level. In this example, assuming that the target probability is 4%, *we can see from the calculations above that actually* $\mathbf{PD}_B = 4\%$.

It is easy to prove that the amount of economic capital to achieve this target is just above 26.75, not 35, so $\mathbf{EC}_1 = 26.75$ *and the remaining is equity capital that could be invested in other risky assets. We suppose that the stockholder require a return on economic capital of* $\pi = 5\%$. *The cost of capital is:*

$$cc_1 = \frac{\frac{26.75}{100}5\%}{1-5\%} = 1.408\%$$

The total ms_1 *to charge on* A_1, *by formula* (10), *will then become*

$$ms_1 = cs_1^* + f_1 + cc_1 = 1.752\% + 1.406\% + 1.408\% = 4.556\%$$

3 Balance Sheet with Two Assets (Uncorrelated Defaults)

We now move on to a multi-period setting, where the bank, after the investment in asset A_1, decides to further invest in a new asset A_2, whose initial price is X_2 and terminal pay-off $A_2(T) = X_2 \times (1 + s_2)$. Without a great loss of generality, we assume that the expiry of the asset A_2 is the same as asset A_1, in T, and that the investment in the new asset occurs in t^+, just an instant after the initial time t; to lighten the notation, we will set $t^+ = t$ in what follow, even though they are two distinct instants.

Also for asset A_2 there is a probability $\mathbf{PD}_2$ that the asset's issuer defaults, in which case the buyer of the asset will receive a stochastic recovery $\mathbf{Rec}_2$ expressed as a percentage of the face value X_2. $\mathbf{Rec}_2$ can take values $\mathbf{Rec}_{2_l}$, for $l = 1, ..., L$, and each possible value can occur with probability p_{2_l}. We assume for the moment that the defaults of the issuers of assets A_1 and A_2 are uncorrelated.

Similarly to asset A_1, for a (non-leveraged) investor

$$A_2(t) = \mathbf{E}\left[X_2 \times (1 + s_2)\right] = X_2 \times (1 + s_2)(1 - \mathbf{PD}_2) + X_2\overline{\mathbf{Rec}}_2\mathbf{PD}_2 = X_2 \qquad (16)$$

where $\overline{\mathbf{Rec}}_2 = \sum_{l=1}^{L} \mathbf{Rec}_{2_l} p_{2_l}$ is the expected recovery in the event of the issuer's bankruptcy. The fair credit spread is derived as n the case of asset A_1 and it is:

$$s_2 = cs_2 = \frac{\overline{\mathbf{Lgd}}_2\mathbf{PD}_2}{1 - \mathbf{PD}_2} \qquad (17)$$

where $\overline{\mathbf{Lgd}}_2 = (1 - \overline{\mathbf{Rec}}_2)$.

Assume now that the bank buys the asset $A_2(t) = X_2$ by issuing new debt: the total debt is $D(t) = D_1(t) + D_2(t) = X_1 + X_2 = X$, *i.e.*: the leverage increases as well. Let f_2 be the funding spread paid on debt $D_2(t)$.

The funding spread requested by bank's bond holders on the new debt $D_2(t)$ is derived in a way similar to equation (3):

$$\begin{aligned} D_2(t) &= \mathbf{E}\left[X_2(1+f_2)\right] \\ &= X_2(1+f_2)(1-\mathbf{PD}_1)(1-\mathbf{PD}_2) \\ &+ \sum_{j=1}^{J} \min\left[\frac{X_1\mathbf{Rec}_{1_j} + X_2(1+s_2) + E}{X}; (1+f_2)\right] X_2 p_{1_j}\mathbf{PD}_1(1-\mathbf{PD}_2) \\ &+ \sum_{l=1}^{L} \min\left[\frac{X_1(1+s_1) + X_2\mathbf{Rec}_{2_l} + E}{X}; (1+f_2)\right] X_2 p_{2_l}\mathbf{PD}_2(1-\mathbf{PD}_1) \\ &+ \sum_{j=1}^{J}\sum_{l=1}^{L} \min\left[\frac{X_1\mathbf{Rec}_{1_j} + X_2\mathbf{Rec}_{2_l} + E}{X}; (1+f_2)\right] X_2 p_{1_j} p_{2_l}\mathbf{PD}_2\mathbf{PD}_1 \\ &= X_2 \end{aligned} \tag{18}$$

The spread s_2 can be either the credit spread or the mark-up spread set by the bank. The funding spread f_2 can be found by solving equation (18): also in this case, as for the case in the previous section for a single asset balance sheet, the solution f_2 is found by a very quick numerical search, starting by setting $f_2 = 0$. It should be noted that in this stylised multi-period setting, the funding spread f_1 on the first debt D_1 enters in the pricing of the new debt D_2 as an input: the bank cannot change this parameter that is contractually set until the expiry in T and this is consistent with the real world. It should be also noted that this spread is no more fair and it should be revised, since the new recovery ratio for the total debt D is now different from the original one set before the investment in the asset A_2. This anyway cannot happen, as the outstanding debt pays a contract rate fixed until the expiry.

The mark-up margin, set by the bank on the second asset, is such that the expected net value of the bank is still the amount of equity posted by shareholders:

$$\begin{aligned} \mathbf{VB}(t) &= \mathbf{E}\left[X_1(ms_1 - f_1) + X_2(ms_2 - f_2) + E\right] \\ &= \left[X_1(ms_1 - f_1) + X_2(ms_2 - f_2) + E\right](1-\mathbf{PD}_1)(1-\mathbf{PD}_2) \\ &+ \sum_{j=1}^{J} \max\left[X_1(\mathbf{Rec}_{1_j} - (1+f_1)) + X_2(ms_2 - f_2) + E; 0\right] p_{1_j}\mathbf{PD}_1(1-\mathbf{PD}_2) \\ &+ \sum_{l=1}^{L} \max\left[X_1(ms_1 - f_1) + X_2(\mathbf{Rec}_{2_l} - (1+f_2)) + E; 0\right] p_{2_l}\mathbf{PD}_2(1-\mathbf{PD}_1) \\ &+ \sum_{j=1}^{J}\sum_{l=1}^{L} \max\left[X_1(\mathbf{Rec}_{1_j} - (1+f_1)) + X_2(\mathbf{Rec}_{2_l} - (1+f_2)) + E; 0\right] p_{1_j} p_{2_l}\mathbf{PD}_2\mathbf{PD}_1 = E \end{aligned} \tag{19}$$

Let

$$\overline{R}_1^* = \sum_{j=1}^{J} \max\left[X_1(\mathbf{Rec}_{1_j} - (1+f_1)) + X_2(ms_2 - f_2) + E; 0\right] p_{1_j},$$

$$\overline{R}_2^* = \sum_{l=1}^{L} \max\left[X_1(ms_1 - f_1) + X_2(\mathbf{Rec}_{2_l} - (1+f_2)) + E; 0\right] p_{2_l}$$

and

$$\overline{R}_{1,2}^* = \sum_{j=1}^{J}\sum_{l=1}^{L} \max\left[X_1(\mathbf{Rec}_{1_j} - (1+f_1)) + X_2(\mathbf{Rec}_{2_l} - (1+f_2)) + E; 0\right] p_{1_j} p_{2_l}.$$

We will rewrite the (19) in a lighter notation:

$$\begin{aligned}\mathbf{VB}(t) =& \mathbf{E}\left[X_1(ms_1 - f_1) + X_2(ms_2 - f_2) + E\right] \\ &= \left[X_1(ms_1 - f_1) + X_2(ms_2 - f_2) + E\right](1 - \mathbf{PD}_1)(1 - \mathbf{PD}_2) \\ &+ \overline{R}_1^*\mathbf{PD}_1(1 - \mathbf{PD}_2) + \overline{R}_2^*\mathbf{PD}_2(1 - \mathbf{PD}_1) \\ &+ \overline{R}_{1,2}^*\mathbf{PD}_2\mathbf{PD}_1 = E\end{aligned} \tag{20}$$

The solution ms_2 to (20) can be found by a quick numerical procedure, by initialising $ms_2 = 0$. It is possible, anyway, to distinguish two cases that will shed some light on which is the value that ms_2 can take.

Before analysing the two distinct cases, we provide the probability of default of the bank once the new asset is bought. As before, we need to check for all the cases when the bank's value drops below zero, in which case the limited shareholders' liability floors the value at zero. We have that :

$$\begin{aligned}\mathbf{PD}_B =& \sum_{j=1}^{J} \mathbf{1}_{\{R_{1_j}^*=0\}} p_{1_j}\mathbf{PD}_1(1 - \mathbf{PD}_2) \\ &+ \sum_{l=1}^{L} \mathbf{1}_{\{R_{2_l}^*=0\}} p_{2_l}\mathbf{PD}_2(1 - \mathbf{PD}_1) \\ &+ \sum_{j=1}^{J}\sum_{l=1}^{L} \mathbf{1}_{\{R_{1_j,2_l}^*=0\}} p_{1_j} p_{2_l}\mathbf{PD}_1\mathbf{PD}_2\end{aligned} \tag{21}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function and $R_{a_b}^*$ is the b-th addend in the summation in each R_a^*.

The Default of the Asset A_2 Does not Imply the Default of the Bank

First, we assume that the default of the asset A_2 does not imply the default of the bank. This may happen for a variety of reasons, but mainly because the quantity X_2 of the asset bought by the bank is small compared to the entire balance sheet. So, even in the event of bankruptcy of the issuer of A_2, the bank is able to cover losses and to repay all its creditors without completely depleting its equity capital E and hence declaring default. On the other hand, the default of the asset A_1 still causes the default of the bank as before.

If X_2 is much smaller than the quantity X_1 of asset A_1 already included in the assets of the bank's balance sheets, then we have the following approximations:

$$\overline{R}_1^* \approx \overline{R}_{1,2}^* \approx \overline{R}_1$$

and

$$\overline{R}_2^* \approx X_1(ms_1 - f_1) + X_2(\overline{\mathbf{Rec}}_2 - (1 + f_2)) + E$$

After substituting the values above, equation (20) can be written as:

$$\begin{aligned}\mathbf{VB}(t) &= \mathbf{E}\left[X_1(ms_1 - f_1) + X_2(ms_2 - f_2) + E\right] \\ &= \left[X_1(ms_1 - f_1)) + X_2(ms_2 - f_2) + E\right](1 - \mathbf{PD}_1)(1 - \mathbf{PD}_2) \\ &+ \left[X_1(ms_1 - f_1) + X_2(\overline{\mathbf{Rec}}_2 - (1 + f_2)) + E\right]\mathbf{PD}_2(1 - \mathbf{PD}_1) \\ &+ \overline{R}_1\mathbf{PD}_1 = E\end{aligned} \tag{22}$$

Let $[(ms_1 - f_1)X_1 + E] = H$; we can re-write (22) as

$$\begin{aligned}\mathbf{VB}(t) = {} & H[(1 - \mathbf{PD}_1)(1 - \mathbf{PD}_2) + \mathbf{PD}_2(1 - \mathbf{PD}_1)] + \overline{R}_1\mathbf{PD}_1 \\ & + \left[X_2(ms_2 - f_2)\right](1 - \mathbf{PD}_1)(1 - \mathbf{PD}_2) \\ & + \left[X_2(\overline{\mathbf{Rec}}_2 - (1 + f_2))\right]\mathbf{PD}_2(1 - \mathbf{PD}_1) = E\end{aligned} \tag{23}$$

By formula (5) and the definition of ms_1:

$$[(ms_1 - f_1)X_1 + E](1 - \mathbf{PD}_1) + \overline{R}_1\mathbf{PD}_1 = E$$

then the first term on the RHS of the first line in (23) is:

$$H[(1 - \mathbf{PD}_1)(1 - \mathbf{PD}_2) + \mathbf{PD}_2(1 - \mathbf{PD}_1)] + \overline{R}_1\mathbf{PD}_1 = H(1 - \mathbf{PD}_1) + \overline{R}_1\mathbf{PD}_1 = E$$

so that:

$$\begin{aligned}\mathbf{VB}(t) = {} & [X_2(ms_2 - f_2)(1 - \mathbf{PD}_1)(1 - \mathbf{PD}_2)] + \\ & [X_2(\overline{\mathbf{Rec}}_2 - (1 + f_2))]\mathbf{PD}_2(1 - \mathbf{PD}_1) = 0\end{aligned} \tag{24}$$

Recalling that $X_2(1 - \overline{\mathbf{Rec}}_2) = X_2\overline{\mathbf{Lgd}}_2$, equation (24) simplifies:

$$\mathbf{VB}(t) = \left[X_2(ms_2 - f_2)(1 - \mathbf{PD}_2) - (X_2\overline{\mathbf{Lgd}}_2 + f_2)\mathbf{PD}_2\right](1 - \mathbf{PD}_1) = 0 \tag{25}$$

which holds if

$$\left[X_2 ms_2(1 - \mathbf{PD}_2) - X_2 f_2 - X_2\overline{\mathbf{Lgd}}_2\mathbf{PD}_2\right] = 0$$

or

$$ms_2 = \frac{\overline{\mathbf{Lgd}}_2\mathbf{PD}_2 + f_2}{1 - \mathbf{PD}_2} = cs_2 + \frac{f_2}{1 - \mathbf{PD}_2} \tag{26}$$

The fair margin spread ms_2 is no more equal to the credit spread cs_2, as it was the case for asset A_1 (see equation (7)). It is worth noting that the total margin spread is made of two components:

- the credit spread cs_2, equal to the spread requested by a non-leveraged investor. This is different from the credit spread required on asset A_1 by the bank, in equation (6). The reason is that when the asset A_1 issuer goes defaulted, this will cause also the the bank's default and the shareholders are liable only up to the amount E to the debt holders, hence a reduced credit spread cs_1^* can be applied; but the default of the issuer of the asset A_2 does not cause the bank's default, so that the loss is fully covered by the equity, so in this case the limitation of the liability up to E is immaterial, and the entire (expected) loss given default $\overline{\mathbf{Lgd}}_2$ must be born by the share holders. As a consequence, the credit spread applied on the asset A_2 in the mark-up margin is equal to cs_1, in the non-leveraged investor case;

- the funding spread f_2, conditioned to the survival of the issuer of the asset A_2. Also in this case there is a difference with the case of the asset A_1: the funding spread in (6) is not divided by the survival probability of the issuer of A_1 since the bank is interested in receiving the amount $f_1 X_1$ without considering the issuer's default. In fact, when the issuer of A_1 goes defaulted, the margin spread m_1 is not cashed in by the bank and it cannot repay the funding costs on the debt D_1. But in the event of default also the bank goes bankrupt, so that shareholders are not interested in the missing payments since they are anyway limited up to the amount E. In case of asset A_2, the missing payment of f_2 has to be covered by the shareholders, which will pay $f_2 X_2$ on the debt D_2 in any case. The expected loss due to this cost must equal the expected return embedded within the mark-up spread ms_2: $\mathbf{E}[f_2^* X_2] = f_2^* X_2(1 - \mathbf{PD}_2) = f_2 X_2$, which means that $f_2^* = f_2/(1 - \mathbf{PD}_2)$.

We can state the following result:

Proposition 3.1. *When pricing an asset that represents a small percentage of the bank's total assets and whose default does not affect the bank's default, the correct (approximated) and theoretically consistent mark-up margin to apply includes the issuer's credit spread, fair to a non-leveraged investor, plus the bank's funding spread conditioned to the issuer's survival probability.*

This result is an accordance with the result in Castagna [4], where the fair spread that an issuer would pay to a non-leveraged investor includes only the credit spread (cs_2 in this case), whereas the mark-up margin charged by the leveraged bank includes also the funding costs (see also chapter 11 in Castagna and Fede [9]). In Castagna [4] the irrelevance of the default of the bank was postulated in an axiom based on the *going concern* principle. In this work we are working without any postulated irrelevance; on the contrary we are explicitly considering the case when such irrelevance arises. An example is given by an asset representing a small percentage of the total bank's assets: when its issuer defaults, this bankruptcy would not affect the survival of the bank. In this case the results are re-derived and confirmed in a much theoretically sounder framework extending the simplified economies in Modigliani& Miller and Merton, thus delimiting the cases when their results remain fully applicable.

The results just shown confirm also that the practice of including the funding valuation adjustment (**FVA**) in the valuation (*i.e.*: internal pricing) process of a contract is fully justified: this thesis was supported in Castagna [7] (arguing against the opposite view in Hull&White [12] and [14]) but not proved analytically. A first attempt to extend the Modigliani&Miller [17] and Merton [16] framework in a multi-period setting was in Castagna [8], but there the simplified structure of the bank's balance sheet did not allow to achieve a fully convincing justification of the all-inclusive pricing rules followed by practitioners. We here showed that the main point is to consider also a richer balance sheet with a more complex set of interrelations between assets' issuers and the bank.

If we consider also the economic capital that the asset A_2 entails to ensure that the default probability of the bank is equal or smaller than given level, the margin spread can be easily modified:

$$ms_1 = \frac{\overline{\mathbf{Lgd}}_2\mathbf{PD}_2 + \mathbf{EC}_2/X_2\pi + f_2}{1-\mathbf{PD}_2} = cs_2^* + \frac{f_2}{1-\mathbf{PD}_2} + cc_2 \tag{27}$$

where $\mathbf{EC}_2$ is the economic capital required by the investment in asset A_2 and $cc_2 = \mathbf{EC}_2/X_2\pi$ is the associated cost.

Example 3.1. *We reprise the example 2.1 and we assume that an instant after the start of the activities in $t = 0$, i.e.: in t' (which, by an approximation, we set equal to t) the bank invests in an asset A_2, whose issuer can default between times 0 and 1 with probability $\mathbf{PD}_2 = 6\%$. There are three recovery scenarios, with associated probability, as shown in the following table:*

	$\overline{\mathbf{Rec}}_{2_l}$	p_l
	75%	*0.2*
	35%	*0.7*
	5%	*0.1*
Exp Rec $\overline{\mathbf{Rec}}_2$	*40%*	

We can immediately compute the fair spread $s_2 = cs_2$ that a non-leveraged investor would require on this asset:

$$s_2 = cs_2 = \frac{60\% \times 5\%}{1-6\%} = 3.83\%$$

The price of the asset in $t' = t = 0$ is $A_2(0) = 10$ and the bank issues new debt for an equivalent amount $D_2(0) = 10$: the amount represents a relatively small fraction of the entire assets held by the bank; moreover, when the issuer of A_2 defaults, the bank will not go bankrupt in any recovery scenario. As in the first example, we suppress all references to time from now on.

The bank's balance sheet will now be:

Assets	***Liabilities***
$B = 35$	$D_1 = 100$
$A_1 = 100$	$D_2 = 10$
$A_2 = 10$	
	————————
	$E = 35$

The spread requested by the creditors of the bank can be found be recursively solving equation (18), starting by setting $f_2 = 0$. We come up with the result:

$$f_2 = 1.318\%$$

The fair spread ms_2 that the bank has to charge on asset A_2, can be found by means of (26)

$$ms_2 = cs_2 + \frac{f_2}{1 - \mathbf{PD}_2} = 3.830\% + \frac{1.318\%}{1 - 6\%} = 5.229\%$$

The simple rule often followed by banks to set the spread (under the hypothesis that interest rates are zero) is to charge in the margin the credit spread referring to the issuer of the asset and the funding cost:

$$ms_2 = 3.830\% + 1.318\% = 5.148\%$$

in case of low issuer's probability of defaults this is an approximation of the fair margin that should be applied.

The probability of default of the bank, after the investment in the asset A_2, is calculated from equation (52) and it is $\mathbf{PD}_B = 4.01\%$. This means that the equity is almost fully sufficient to keep the probability of default at the chosen level of 4% we mentioned in example 2.1. To avoid unneeded complication, we take the new $\mathbf{PD}_B$ as compliant with the limit in practice, which means that the economic capital absorbed by the new asset is $\mathbf{EC} = 8.25$. If we want to include also the remuneration for $\mathbf{EC}_2$ in the margin, then from (27) we have:

$$cc_2 = \frac{\frac{8.25}{10} 5\%}{1 - 6\%} = 4.145\%$$

so that the total margin is $ms_2 = 5.148\% + 4.125\% = 9.273\%$. In this example the economic capital requested to keep $\mathbf{PD}_B \leq 4\%$ is equal to amount invested in A_2: this is due to the simplified balance sheet we are considered and to the small number of scenarios for the recovery of the two assets. In the real world, in a much more complex balance sheet, the economic capital for an additional investment is typically smaller than the invested amount and consequently also the cost of capital component of the margin is much smaller.

The Default of the Asset A_2 Implies the Default of the Bank

Assume now that the default of the asset A_2 implies the default of the bank: this is typically the case when the asset A_2 is, in percentage terms, a great share of the bank's total assets. The margin spread that the bank has to apply on the second asset is derived numerically by solving equation (23). It is possible to write down a formally closed form formula, to assess which are the parameters affecting the spread.

By rewriting equation (20), we have:

$$\begin{aligned} \mathbf{VB}(t) &= \mathbf{E}\left[X_1(ms_1 - f_1) + X_2(ms_2 - f_2) + E\right] \\ &= \left[X_1(ms_1 - f_1) + X_2(ms_2 - f_2) + E\right](1 - \mathbf{PD}_1)(1 - \mathbf{PD}_2) + \epsilon \\ &= E \end{aligned} \tag{28}$$

where $\epsilon = \overline{R}_1^* \mathbf{PD}_1(1-\mathbf{PD}_2) + \overline{R}_2^* \mathbf{PD}_2(1-\mathbf{PD}_1) + \overline{R}_{1,2}^* \mathbf{PD}_1\mathbf{PD}_2$. Let $\mathbf{PD}_{1,2} = 1-(1-\mathbf{PD}_1)(1-\mathbf{PD}_2)$: we solve for ms_2 and we get:

$$
\begin{aligned}
ms_2 &= \frac{\frac{E-(1-\mathbf{PD}_1)(1-\mathbf{PD}_2)[X_1(ms_1-f_1)+E]}{X_2} - \epsilon}{(1-\mathbf{PD}_1)(1-\mathbf{PD}_2)} + f_2 \\
&= cs_2^* + f_2
\end{aligned}
\tag{29}
$$

The margin spread ms_2 resembles the margin spread ms_1 derived for asset A_1: it is made of three constituent parts:

- an "adjusted" credit spread cs_2^*, due to the limited liability of the shareholders, referring to the asset A_2: the loss given default suffered by the bank, E/X_2, is smaller than the loss given default of the asset A_2, $\mathbf{Lgd}_2$, suffered by a non-leveraged investor. This lower loss given default multiplies the probability that either the issuer of the asset A_2 or the issuer of asset A_1 goes bust, to get the expected loss, and then divided by the survival probability of the bank (*i.e.*: the joint probability that both issuers of assets A_1 and A_2 survive). This means that the margin spread should equal the expected loss given default, given the bank's survival;

- the funding spread f_2 paid by the bank on the new debt issued to buy the asset A_2. The funding spread is not divided by the survival probability of issuer since when it defaults, the bank will go bankrupt as well and it will not pay the spread to its creditors.

It is useful to check how much the limited liability of the shareholder, in case of the default of the issuer of A_2, impacts on the final margin spread with respect to the case analysed before, when the issuers default did not trigger the banks default. Let ms_2^{ND} be the spread in case the amount of A_2 is marginal with respect the entire amount of assets held by the bank: in this case we know from before that $ms_2^{ND} = cs_2 + f_2/(1-\mathbf{PD}_2)$. We now introduce a new quantity, which we tentatively name Limited Liability Valuation Adjustment (**LLVA**). This quantity is the correction to the spread ms_2 charged for the investment in A_2 when its amount increases so that it is less and less marginal with respect to the other assets; this means that the default of the issuer of the asset A_2 can trigger also the bank's default and the limited liability of the shareholders up to the existing equity E will come into play.

We formally define the quantity **LLVA** for the asset A_2 as:

$$
\mathbf{LLVA} = ms_2 - ms_2^{ND} = ms_2 - cs_2 + f_2/(1-\mathbf{PD}_2) \tag{30}
$$

so, the general margin spread can be expressed as:

$$
ms_2 = c_2 + \frac{f_2}{1-\mathbf{PD}_2} + \mathbf{LLVA} \tag{31}
$$

Even if the capital markets are perfect, the total margin spread m_2 differs from the market credit spread cs_2 and it will typically be greater than the latter. When the amount of asset A_2 held by the bank increases, the Limited Liability Valuation Adjustment (**LLVA**) tends to abate the margin ms_2 and it can be also lower than cs_2 for sufficiently large A_2. In any case, ms_2 will no more be the sum of the credit spread cs_2 and the bank's funding spread f_2, as in the case above when the default of the asset A_2 did not imply the bank's default.

Proposition 3.2. *When evaluating an asset whose default implies the bank's default, the correct and theoretically consistent mark-up margin to apply includes the issuer's credit spread, the funding spread (divided by the survival probability of the issuer) paid on the new debt to buy the asset, and an adjustment due to the limited liability of the shareholders.*

Example 3.2. *We continue from the example 2.1 and we assume that the bank wishes to invest in A_2 an amount that is big enough to trigger its own default when the issuer goes bust, in at least one recovery scenario. In this case formula* (29) *is computed by numerically solving* (20) *by a numerical procedure.*

In table 2 we show the effect of increasing the investment in A_2 by issuing new debt. The probability of default will increase because the shareholder would pour more equity in the bank to preserve the desired target level (it was set at 4% before). Consequently, also the funding spread f_2, after an initial decrease due to the larger amount of assets on which they can try to recover the credit exposure on a default event, eventually increases since the expected losses suffered by banks creditors will be higher due to the higher $\mathbf{PD}_B$. The Limited Liability Valuation Adjustment will be quite negligible for relatively small amount of A_2, thus we have a confirmation that for small amounts of A_2, (as $X_2 = 10$ of the example before) formula (26) *is a good approximation, since we are getting the same result* 5.23%. *For larger amounts of A_2 the benefit becomes larger, thus lowering the margin ms_2 applied on the asset. The margin will approach the level required by a non-leveraged investor (for $A_2 = 100$). The* **LLVA** *starts being substantial when the amount of $X_2 > 40$, or when the default of the issuer of A_2 implies, in at least one recovery scenario, the default of the bank.*

A_2	f_2	ms_2	**LLVA**	$\mathbf{PD}_B$
10	1.318%	5.321%	-0.001%	4.01%
20	1.215%	5.108%	-0.014%	4.05%
30	1.128%	5.007%	-0.022%	4.05%
40	1.059%	4.904%	-0.053%	4.63%
50	1.038%	4.762%	-0.171%	4.63%
60	1.135%	4.570%	-0.467%	8.62%
70	1.281%	4.274%	-0.918%	8.62%
80	1.411%	4.073%	-1.258%	8.62%
90	1.527%	3.932%	-1.522%	8.62%
100	1.632%	3.832%	-1.735%	8.62%

Table 2: Funding spread, margin, diversification benefit and bank's probability of default as the amount invested in asset A_2 increases.

4 Balance Sheet with Two Assets (Perfectly Correlated Defaults)

Assume now that the defaults of the issuers for assets A_1 and A_2 are perfectly correlated: let $\mathbf{1}_{A_1}$ the indicator function equal to 1 when the default of the issuer of A_1 occurs, and

$\mathbf{1}_{A_2}$ similarly defined. Moreover let **PD** be the probability of default of both issuers. We have:

$$\begin{aligned}
&\mathbf{Pr}[\mathbf{1}_{A_1} = 1 \cap \mathbf{1}_{A_2} = 1] = \mathbf{PD} \\
&\mathbf{Pr}[\mathbf{1}_{A_1} = 0 \cap \mathbf{1}_{A_2} = 0] = 1 - \mathbf{PD} \\
&\mathbf{Pr}[\mathbf{1}_{A_1} = 1 \cap \mathbf{1}_{A_2} = 0] = \mathbf{Pr}[\mathbf{1}_{A_1} = 0 \cap \mathbf{1}_{A_2} = 1] = 0
\end{aligned}$$

The fair funding spread requested by the creditor of the bank on the new debt D_2, in a perfect market, is derived by solving the following equation:

$$\begin{aligned}
D_2(t) &= \mathbf{E}\left[X_2(1+f_2)\right] \\
&= X_2(1+f_2)(1-\mathbf{PD}) \\
&+ \sum_{j=1}^{J}\sum_{l=1}^{L} \min\left[\frac{X_1\mathbf{Rec}_{1_j} + X_2\mathbf{Rec}_{2_l} + E}{X}; (1+f_2)\right] X_2 p_{1_j} p_{2_l}\mathbf{PD} \\
&= X_2
\end{aligned} \tag{32}$$

The margin the bank has to apply on the asset A_2, provided it has enough bargaining power, is obtained by a modified version of equation (33) that accounts for the perfect correlation between the issuers of the two assets:

$$\begin{aligned}
\mathbf{VB}(t) =& \mathbf{E}\left[X_1(ms_1 - f_1) + X_2(ms_2 - f_2) + E\right] \\
&= \left[X_1(ms_1 - f_1) + X_2(ms_2 - f_2) + E\right](1-\mathbf{PD}) \\
&+ \overline{R}^{*}_{1,2}\mathbf{PD} = E
\end{aligned} \tag{33}$$

where the notation is the same used before.

If we further assume that the bankruptcy of the two issuers triggers, in at least in one of the possible cases of loss given default, the default of the bank then the formula for the fair spread ms_2 can be formally written as:

$$\begin{aligned}
ms_2 &= \frac{\frac{E\mathbf{PD} + X_1(ms_1 - f_1)(1-\mathbf{PD}) - \overline{R}^{*}_{1,2}\mathbf{PD}}{X_2}\mathbf{PD}}{(1-\mathbf{PD})} + f_2 - \frac{\frac{X_1(ms_1 - f_1)(1-\mathbf{PD}) + \overline{R}^{*}_{1,2}1\mathbf{PD}}{X_2}}{(1-\mathbf{PD})} \\
&= cs_2^{**} + f_2
\end{aligned} \tag{34}$$

Formula (34) is similar to (29) for the case when default are correlated. The main difference is that the joint probability of default of the two issuers $(1-\mathbf{PD}_1)(1-\mathbf{PD}_2)$ has been replaced by $(1-\mathbf{PD})$, the probability of the issuer of asset A_2 is **PD**. Obviously also in this case we can express the margin as $ms_2 = cs_2 + \frac{f_2}{1-\mathbf{PD}_2} + \mathbf{LLVA}$; the **LLVA** will take into account the perfect correlation of the two defaults.

5 Balance Sheet with a Derivative Contract

We have analysed how to bank should evaluate (*i.e.*: internally price) assets when it decides to invest in them. Assume now that the bank, after investing at time $t = 0$ in asset A_1, enters in a derivative contract, instead of buying another asset. Actually, entering in a contract can be seen as buying an asset or issuing a liability, or in some cases doing both depending on contingent evolution of (typically financial) variables (as, for example, when a swap contract is closed). In this case, the valuation of the contract can be operated as it were an asset (or a liability) with stochastic intermediate cash flows and terminal pay-off, whose expected amounts are discounted at the reference date.

We would like here to stress that a bank can also offer a service of market-making for derivative contracts: in this case it does not really try to buy an asset or issue a liability; on the contrary, the bank is selling a product manufactured with a given technology. For example, the bank can be a market-maker for options on a given asset: it is not really interested in buying or selling options based on some expectation of the future evolution of the underlying asset, simply it sells a product that can be a long or a short position in the option. Once one of two positions is sold to the client, the bank has the internal skills to use the available technology to manufacture the product so that its profit derives from the ability to sell at a margin over the production cost.[4]

The Black&Scholes (B&S) model, for example, is a technology to manufacture (or replicate, to use the financial term) an option contract. More generally, the replication hinges on the idea to set up a (possibly continuously rebalancing) trading strategy that satisfies the following conditions:

1. Self financing condition, that is: no other investment is required in operating the strategy besides the initial one;

2. Replicating condition, that is: at any time t the replicating portfolio's value equals the value of the contract.

Keeping the analysis in a discrete time setting, we work in the classical binomial setting by Cox and Rubinstein [10]. Assume that the underlying asset is $S(t) = S$ at time t, and that it can go up to $S_u = Su$ or down to $S_d = Sd$, with $d < 1$, $u > 1$ and $u \times d = 1$ in next period T, with a probability, respectively, equal to q and $1 - q$. Let $V(t)$ be the price of a derivative contract at time t, and V_u and V_d its value when the underlying jumps to, respectively, to S_u and S_d. Besides, let $B(t) = B$ be the value in t of a risk-free bank-account deposit earning the risk-free rate. We want to replicate a long position in the derivative contract.

To do so, we have to set the following equalities in each of the two state of the world (*i.e.*: possible outcomes of the underlying asset's price):

$$V_u = \alpha Su + \beta B(1 + r) \tag{35}$$

and

$$V_d = \alpha Sd + \beta B(1 + r) \tag{36}$$

Equations (35) and (36) are a system that can be easily solved for quantities α and β, yielding:

$$\alpha = \Delta = \frac{V_u - V_d}{(u - d)S} \tag{37}$$

[4] In Castagna [8] we adopt this "industrial" metaphor in a rather discursive argument to justify the inclusion of funding cost in the internal evaluation of contracts by the bank. In this work we will expand in a more quantitative way the ideas presented in there.

and

$$\beta = \frac{uV_d - dV_u}{(u-d)B(1+r)} \tag{38}$$

We have indicated $\alpha = \Delta$ because it is easily seen in (37) that it is the numerical first derivative of the price of the contingent claim with respect to the underlying asset, usually indicated so in the Option Pricing Theory.

If the replicating portfolio is able to mimic the pay-off of the contract, then its value at time t is also the arbitrage-free price of the contract:

$$V(t) = \Delta S + \beta B = \frac{V_u - V_d}{(u-d)} + \frac{uV_d - dV_u}{(u-d)(1+r)} \tag{39}$$

It is possible to express (39) in terms of discounted expected value under the risk neutral measure:

$$V(t) = \frac{1}{1+r}[pV_u + (1-p)V_d] \tag{40}$$

with $p = \frac{(1+r)-d}{u-d}$. $V(t)$ is the expected risk neutral terminal value of the contract, discounted with the risk-free rate.

This is the original setting by Cox and Rubinstein [10] that assumed a perfect and friction-less market, where default risk is absent (hence, the bank cannot go bankrupt and there are no differential rates for borrowing or lending). But if the bank is a defaultable agent, it is no more guaranteed that the replication argument works exactly as described above. We try to investigate this matter now.

Let us go back to our setting above, after the bank started its activities and invested in asset A_1, at time t'. As before, for sake of simplicity of the notation, we set $t' = t$ in practice, even if they are two distinct instants. At this time, the bank closes a derivative contract denoted by ϕV, where $\phi = \pm 1$ depending on whether the bank has, respectively, a long or short position in it. The replication strategy involves trading in a quantity $-\phi\alpha$ in the underlying asset S and $-\phi\beta$ in a bank deposit: in the real world being long a risk-free bank account means for the bank lending money to a risk-free counterparty[5] whereas being short means for the bank borrowing money.

The standard replication argument does not consider the default of the replicator (the bank) so that borrowing and lending money occurs at the same interest rate. In reality, the replicator can borrow money at a rate which possibly includes a spread for the risk of its own default: this is the funding spread that we analysed above.[6] In the following analysis we assume that the default of the counterparty of the derivative contract (occurring with probability $\mathbf{PD}_V$) is independent from the evolution of the underlying asset and that

[5]It can be alternatively assumed that money is lent to a defaultable counterparty, at a rate including a credit spread compensating for the risk. In this case the expected interest rate yielded by the loan is the risk-free rate, when accounting for the expected losses given default.

[6]See also Castagna [6] and Castagna and Fede [9] for more details on the replication of contract including funding and liquidity costs. For an extended formal treatment of the replication of a contract in a world with differential rates, see Mercurio [15].

the issuer of the latter cannot default. The possible recoveries of the final pay-off of the derivative contract are $\mathbf{Rec}_{V_l}$, for $l = 1, ..., L$, each with occurring with probability p_{V_l}.

First we need to know if the bank is able to borrow money (if this is prescribed by the replication strategy) at the risk-free rate, as supposed by the theory. If the bank needs to borrow money (*i.e.*: $-\phi\beta < 0$), it will issue new debt $D_2 = |\phi\beta B|$: in a market with perfect information the creditors will set a credit (funding, from bank's perspective) spread to remunerate the credit risk. If the bank has to invest positive cash-flows in a risk-free bank account (*i.e.*: $-\phi\beta > 0$), it will earn the risk-free rate (set equal to 0 in the current analysis).

The credit spread required by the creditor of the bank (alternatively said, the funding spread f_2 the bank has to pay) is determined by the following equation (recall we are setting $r = 0$, the extension to non-zero rates is straightforward):

$$
\begin{aligned}
&D_2(t) = \mathbf{E}\left[|\phi\beta B|(1+f_2)\right] \\
&= |\phi\beta B|(1+f_2)(1-\mathbf{PD}_1)(1-\mathbf{PD}_V) \\
&+ \sum_{j=1}^{J} \min\left[\frac{X_1\mathbf{Rec}_{1_j} + \phi(\overline{V} - \Delta\overline{S}) + E}{X_1 + |\phi\beta B|}; (1+f_2)\right] |\phi\beta B|\, p_{1_j}\mathbf{PD}_1(1-\mathbf{PD}_V) \\
&+ \sum_{l=1}^{L} \min\left[\frac{X_1(1+s_1) + \mathbf{EPE}_V\mathbf{Rec}_{V_l} - \phi\Delta\overline{S} + E}{X_1 + |\phi\beta B|}; (1+f_2)\right] |\phi\beta B|\, p_{2_l}\mathbf{PD}_V(1-\mathbf{PD}_1) \\
&+ \sum_{j=1}^{J}\sum_{l=1}^{L} \min\left[\frac{X_1\mathbf{Rec}_{1_j} + \mathbf{EPE}_V\mathbf{Rec}_{V_l} - \phi\Delta\overline{S} + E}{X_1 + |\phi\beta B|}; (1+f_2)\right] |\phi\beta B|\, p_{1_j}p_{V_l}\mathbf{PD}_V\mathbf{PD}_1 \\
&= |\phi\beta B|
\end{aligned}
\tag{41}
$$

where $\mathbf{EPE}_V = \phi V^+(T) = \phi[pV_u\mathbf{1}_{\{\phi V_u>0\}} + (1-p)V_d\mathbf{1}_{\{\phi V_d>0\}}]$ is the expected positive exposure (the bank looses on the counterparty's default only if the contract has a positive value); $\overline{V} = \mathbf{E}[V]$ and $\overline{S} = \mathbf{E}[S]$. Basically the first second line of equation (41) is the amount the creditors of the bank receive if the bank survives (*i.e.*: if no default of the issuer of asset A_1 or counterparty of the derivative contract) occurs; the third, fourth and fifth lines is the amount that the creditors of the bank expect to recover if, respectively, the issuer of asset A_1, the counterparty of the derivative contract or both default. The total expected amount returned to the creditors equals the present amount borrowed by the bank, in the last line.

As we have seen above when adding a new asset A_2, if we focus on the case when the new contract is such that the default of the counterparty does not imply the default of the bank, than we should be able to exclude also the cases when the residual value of the assets is negative. Although this cannot happen when replicating plain vanilla options, it cannot be excluded for some exotic options. We will not deal with this very specific situations, but they can be dealt with by introducing a floor at zero in the arguments of the min[] functions.

It should be also manifest that the funding spread f_2 is above zero for non trivial cases, even assuming a perfect market. This is due to the fact that the composition of the existing balance sheet, when closing the contract and starting its replication, does affect the ability of the bank to borrow money at a given spread. Although theoretically the bank is adding a risk-free operation in its balance sheet (*i.e.*: a derivative contract and its associated replication strategy[7]), the funding spread is not nil because the spread on the already issued debt D_1 is fixed and it cannot be updated. Burgard and Kjaer [3] theoretically justify a funding spread equal to zero in the replication strategy by showing that, in a simplified balance sheet, the total spread paid after the start of the replication strategy would be such that it implies a risk-free rate for the incremental debt needed in it. A similar view has been proposed also in Nauta [19]. Nonetheless, this justification relies on the fact that all the outstanding debt is renewed at the moment of the replication's inception, and in any case every time a change in the total outstanding amount occurs: in this way the overall lowering of the bank's riskiness is redistributed on the old and new debt, so that the incremental funding spread would be zero (*i.e.*: the new debt marginally costs the risk-free rate). But, if the existing debt cannot be freely renewed, because it has its own expiry (in T in our setting) and the contract funding spread cannot be changed, then the marginal funding spread on the new debt will not be the risk-free rate.

Hull&White [12], [14] and [13] also think that the inclusion of funding costs is not justified: their argument hinges on the M&M theorem and, in the more recent works, on some example reproducing possible real cases of derivative pricing. We showed above that the funding costs do exist when some of the assumptions made by M&M are not matched in the real world, namely when considering a complex banking activity in a multi-period setting; as such funding costs have to considered, as it will be apparent in what follows. Besides, the authors seem to be more worried about the fact the funding can be seen also as a benefit in the price of the derivative contracts: this may originate some arbitrage opportunities. We will show that we tend to agree with them as far as the inclusion of funding benefit is concerned, but we disagree with them on that funding cost should not be included as well. But more on this later on.

After having determined the funding costs that the creditors of the bank would rationally set on the debt issued by the bank to set up the replication strategy, let us see how this modifies, with respect to the framework outlined above, when we know that the bank needs to borrow money (otherwise no change applies). Equations (35) and (36) can be generalised as follows:

$$V_u = \alpha S u + \beta^F B(1 + r + f_2 \mathbf{1}_{\{\beta^F<0\}}) \tag{42}$$

and

$$V_d = \alpha S d + \beta^F B(1 + r + f_2 \mathbf{1}_{\{\beta^F<0\}}) \tag{43}$$

The funding spread appears only if the bank need to borrow money ($\beta^F < 0$).

Equations (35) and (36) are a system that can be easily solved for quantities α and β^F, yielding:

$$\alpha = \Delta = \frac{V_u - V_d}{(u-d)S} \tag{44}$$

[7]We exclude here the model risk inherent to the replication strategy: the model can be actually a partially correct representation of the reality, so that the replication is not perfect. An example of model risk can be using the B&S model in a world where the underlying asset follows a dynamics with stochastic volatility instead of a deterministic one.

and

$$\beta^F = \frac{uV_d - dV_u}{(u-d)B(1+r+f_2\mathbf{1}_{\{\beta^F<0\}})} \tag{45}$$

The value at time t of the contract is then:

$$V(t) = \Delta S + \beta^F B = \frac{V_u - V_d}{(u-d)} + \frac{uV_d - dV_u}{(u-d)(1+r+f_2\mathbf{1}_{\{\beta^F<0\}})} \tag{46}$$

After some manipulations, it is possible to express (46) in terms of discounted expected value under the risk neutral measure:

$$V^F(t) = \frac{1}{1+r}[pV_u + (1-p)V_d] + \mathbf{FVA} \tag{47}$$

where **FVA** the funding value adjustment, equal to:

$$\mathbf{FVA} = B[\beta^F - \beta] \tag{48}$$

So the value of the contract including funding is equal to the otherwise identical contract valued in an economy where the counterparty risk is excluded (and funding costs are nil) plus the funding value adjustment. When $\beta^F > 0$, then $\beta^F = \beta$ and $\mathbf{FVA} = 0$, otherwise $|\beta| > |\beta^F|$ (since f_2 is a positive quantity) and the **FVA** is a positive quantity as well. This means that a bank that wishes to replicate a long position after going short in the contract, its replication cost is higher than the cost paid in a perfect market where default risk is excluded and $f_2 = 0$ always.

It is worth noting that if the bank wishes to replicate as short position in the contract, after entering a long position with the counterparty, then the contract is worth, in absolute value:

$$|V(t)^F| = \left| -\frac{1}{1+r}[pV_u + (1-p)V_d] + \mathbf{FVA} \right|$$

where **FVA** is defined as before with β^F modified as

$$\beta^F = \frac{uV_d - dV_u}{(u-d)B(1+r+f_2\mathbf{1}_{\{-\beta^F<0\}})}$$

FVA, for the same considerations above, is still a positive number. This means that the short replication cost entails a lower value for the long position in the contract for the replicator. We can summarise these results and write the value to the replicator for a long/short position in the contract as:

$$\phi V^F(t) = \phi\frac{1}{1+r}[pV_u + (1-p)V_d] - \mathbf{FVA} = \phi V(t) - \mathbf{FVA} \tag{49}$$

where $\phi = 1$ if the replicator is long the contract and $\phi = -1$ if it is short. Equation (49) implies a buy/sell price that the replicator would quote (even in absence of bid/ask spreads), since the value when going long the contract is lower then the theoretical risk-free price (equation (40)), while the value when going short is higher in absolute terms (*i.e.*: more negative) then the same theoretical risk-free price. In the Option Pricing Theory no-arbitrage bounds are identified typically by transaction costs, such as pure bid/ask spreads. We are here adding another component widening these bounds, which is given by the funding costs due to the default-riskiness of the replicator (see also Castagna [5], in the final remarks, and Castagna&Fede [9], chapter 10, for a discussion on this topic).

Let us now see how the contract is evaluated by the bank. Let $\Pi(t) = \phi[V^F(t) - \alpha S(t) - \beta^F B]$: if the bank correctly applies the replication strategy prescribed by the model, given that this is a perfect representation of the real world, then we should have that $\Pi(s) = 0$ for any $s \in [t, T]$. We are supposing here that the bank is considering also the funding costs in the replication strategy. Nonetheless, taking into account also the arguments of those that tend to exclude the **FVA**, we want to see what happens when the contract is inserted in the bank's balance sheet and ascertain whether the marginal funding cost is actually the risk-free rate. The evaluation process relies on the following equation:

$$
\begin{aligned}
\mathbf{VB}(t) &= \mathbf{E}\left[X_1(ms_1 - f_1) + \Pi(T) + E\right] \\
&= \left[X_1(ms_1 - f_1) + \overline{\Pi} + E\right](1 - \mathbf{PD}_1)(1 - \mathbf{PD}_V) \\
&+ \sum_{j=1}^{J} \max\left[X_1(\mathbf{Rec}_{1_j} - (1 + f_1)) + \overline{\Pi} + E; 0\right] p_{1_j}\mathbf{PD}_1(1 - \mathbf{PD}_V) \\
&+ \sum_{l=1}^{L} \max[X_1(ms_1 - f_1) + \mathbf{Rec}_{V_l}\mathbf{EPE}_V + \mathbf{ENE}_V \\
&\quad - \phi(\Delta\overline{S} - \beta^F B(1 + f_2\mathbf{1}_{\{-\phi\beta^F<0\}})) + E; 0]p_{2_l}\mathbf{PD}_V(1 - \mathbf{PD}_1) \\
&+ \sum_{j=1}^{J}\sum_{l=1}^{L} \max[X_1(\mathbf{Rec}_{1_j} - (1 + f_1)) + \mathbf{Rec}_{V_l}\mathbf{EPE}_V + \mathbf{ENE}_V \\
&\quad - \phi(\Delta\overline{S} - \beta^F B(1 + f_2\mathbf{1}_{\{-\phi\beta^F<0\}})) + E; 0]p_{1_j}p_{V_l}\mathbf{PD}_V\mathbf{PD}_1 \\
&= E
\end{aligned}
\tag{50}
$$

where $\overline{\Pi} = \mathbf{E}[\Pi(T)]$, $\overline{S} = \mathbf{E}[S]$ and $\mathbf{ENE}_V = \phi V^-(T) = \phi[pV_u\mathbf{1}_{\{\phi V_u<0\}} + (1-p)V_d\mathbf{1}_{\{\phi V_d<0\}}]$ is the expected negative exposure. The loss is computed only on the $\mathbf{EPE}_V$, otherwise it is nil: the bank is unaffected by the counterparty's default when the value of the contract is negative and the expected value of the contract fully enters in the equation.

Let

$$
\overline{R}_1^* = \sum_{j=1}^{J} \max\left[X_1(\mathbf{Rec}_{1_j} - (1 + f_1)) + \overline{\Pi} + E; 0\right] p_{1_j},
$$

$$
\begin{aligned}
\overline{R}_2^* = \sum_{l=1}^{L} &\max[X_1(ms_1 - f_1) + \mathbf{Rec}_{V_l}\mathbf{EPE}_V + \mathbf{ENE}_V \\
&- \phi(\Delta\overline{S} + \beta^F B(1 + f_2\mathbf{1}_{\{-\phi\beta^F<0\}})) + E; 0]p_{V_l}
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{R}_{1,2}^* = \sum_{j=1}^{J}\sum_{l=1}^{L} &\max[X_1(\mathbf{Rec}_{1_j} - (1 + f_1)) + \mathbf{Rec}_{V_l}\mathbf{EPE}_V + \mathbf{ENE}_V \\
&- \phi(\Delta\overline{S} + \beta^F B(1 + f_2\mathbf{1}_{\{-\phi\beta^F<0\}})) + E; 0]p_{1_j}p_{V_l}.
\end{aligned}
$$

We rewrite the (50) in a lighter notation:

$$\begin{aligned}\mathbf{VB}(t) =& \mathbf{E}\left[X_1(ms_1 - f_1) + \Pi(T) + E\right] \\ &= \left[X_1(ms_1 - f_1) + \overline{\Pi} + E\right](1 - \mathbf{PD}_1)(1 - \mathbf{PD}_V) \\ &+ \overline{R}_1^*\mathbf{PD}_1(1 - \mathbf{PD}_V) + \overline{R}_2^*\mathbf{PD}_V(1 - \mathbf{PD}_1) \\ &+ \overline{R}_{1,2}^*\mathbf{PD}_V\mathbf{PD}_1 = E\end{aligned} \tag{51}$$

The probability of default of the bank once the derivative contract is closed, can be computed by considering all all the cases when the bank's value drops below zero, in which case the limited shareholders' liability floors the value at zero. We have that :

$$\begin{aligned}\mathbf{PD}_B = &\sum_{j=1}^{J}\mathbf{1}_{\{R_{1_j}^*=0\}}p_{1_j}\mathbf{PD}_1(1 - \mathbf{PD}_V) \\ &+ \sum_{l=1}^{L}\mathbf{1}_{\{R_{2_l}^*=0\}}p_{2_l}\mathbf{PD}_V(1 - \mathbf{PD}_1) \\ &+ \sum_{j=1}^{J}\sum_{l=1}^{L}\mathbf{1}_{\{R_{1_j,2_l}^*=0\}}p_{1_j}p_{2_l}\mathbf{PD}_1\mathbf{PD}_V\end{aligned} \tag{52}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function and $R_{a_b}^*$ is the b-th addend in the summation in each R_a^*.

The Default of the Counterparty Does not Imply the Default of the Bank

If the notional of the derivative contract is is much smaller than the asset A_1 already included in the assets of the bank's balance sheets, then (analogously to what have seen above for the asset A_2) we have that:

$$\overline{R}_1^* \approx \overline{R}_{1,2}^* \approx \overline{R}_1$$

and

$$\overline{R}_2^* \approx X_1(ms_1 - f_1) + \overline{\mathbf{Rec}}_{V_l}\mathbf{EPE}_V + \mathbf{ENE}_V - \phi(\Delta\overline{S} + \beta^F B(1 + f_2\mathbf{1}_{\{-\phi\beta^F<0\}}) + E$$

The notation is the same as above.

After substituting these values, equation (51) can be written as:

$$\begin{aligned}\mathbf{VB}(t) &= \mathbf{E}\left[X_1(ms_1 - f_1) + \Pi(T) + E\right] \\ &= \left[X_1(ms_1 - f_1)) + \overline{\Pi} + E\right](1 - \mathbf{PD}_1)(1 - \mathbf{PD}_V) \\ &\quad + [X_1(ms_1 - f_1) + \overline{\mathbf{Rec}}_{V_l}\mathbf{EPE}_V + \mathbf{ENE}_V \\ &- \phi(\Delta\overline{S} + \beta^F B(1 + f_2\mathbf{1}_{\{-\phi\beta^F<0\}})) + E]\mathbf{PD}_V(1 - \mathbf{PD}_1) \\ &+ \overline{R}_1\mathbf{PD}_1 = E\end{aligned} \tag{53}$$

By repeating the same reasoning we have presented in the case of two assets, let $[(ms_1 - f_1)X_1 + E] = H$ so that equation (53) can be written

$$\begin{aligned}\mathbf{VB}(t) = {} & H[(1-\mathbf{PD}_1)(1-\mathbf{PD}_V) + \mathbf{PD}_V(1-\mathbf{PD}_1)] + \overline{R}_1\mathbf{PD}_1 \\ & + \overline{\Pi}(1-\mathbf{PD}_1)(1-\mathbf{PD}_V) \\ & + [\overline{\mathbf{Rec}}_{V_l}\mathbf{EPE}_V + \mathbf{ENE}_V - \phi(\Delta\overline{S} + \beta^F B(1 + f_2\mathbf{1}_{\{-\phi\beta^F<0\}}))]\mathbf{PD}_V(1-\mathbf{PD}_1) = E\end{aligned} \tag{54}$$

By formula (5) and the definition of ms_1:

$$[(ms_1 - f_1)X_1 + E](1-\mathbf{PD}_1) + \overline{R}_1\mathbf{PD}_1 = E$$

The first term on the RHS of the first line in (54) is (by equation (5) and the definition of ms_1):

$$H[(1-\mathbf{PD}_1)(1-\mathbf{PD}_V) + \mathbf{PD}_V(1-\mathbf{PD}_1)] + \overline{R}_1\mathbf{PD}_1 = H(1-\mathbf{PD}_1) + \overline{R}_1\mathbf{PD}_1 = E$$

So that:

$$\begin{aligned}\mathbf{VB}(t) = {} & \overline{\Pi}(1-\mathbf{PD}_1)(1-\mathbf{PD}_V) + \\ & + [\overline{\mathbf{Rec}}_{V_l}\mathbf{EPE}_V + \mathbf{ENE}_V - \phi(\Delta\overline{S} + \beta^F B(1 + f_2\mathbf{1}_{\{-\phi\beta^F<0\}}))]\mathbf{PD}_V(1-\mathbf{PD}_1) = 0\end{aligned} \tag{55}$$

or:

$$\begin{aligned}\mathbf{VB}(t) = {} & \big[\overline{\Pi}(1-\mathbf{PD}_V) + [\overline{\mathbf{Rec}}_{V_l}\mathbf{EPE}_V + \mathbf{ENE}_V \\ & - \phi(\Delta\overline{S} + \beta^F B(1 + f_2\mathbf{1}_{\{-\phi\beta^F<0\}}))]\mathbf{PD}_V\big](1-\mathbf{PD}_1) = 0\end{aligned} \tag{56}$$

Since $\overline{\Pi} = \phi\overline{V} - \phi\Delta\overline{S} - \phi\beta^F B(1 + f_2\mathbf{1}_{\{-\phi\beta^F<0\}})$, and $\phi\overline{V} = \phi V^+(T) + \phi V^-(T)$,[8] equation simplifies as:

$$\mathbf{VB}(t) = \left[\overline{\Pi} - \overline{\mathbf{Lgd}}_{V_l}\mathbf{EPE}_V\mathbf{PD}_V\right](1-\mathbf{PD}_1) = 0 \tag{57}$$

where we have set

$$-\overline{\mathbf{Lgd}}_{V_l}\mathbf{EPE}_V = \overline{\mathbf{Rec}}_{V_l}\mathbf{EPE}_V - \phi V^+(T)$$

It is worth noting that by the definition of credit value adjustment:

$$\overline{\mathbf{Lgd}}_{V_l}\mathbf{EPE}_V\mathbf{PD}_V = \mathbf{CVA}$$

We also know that by definition of the replication strategy $\overline{\Pi} = 0$ so that (57) holds only if we add an additional sum of cash at the inception such that we set the initial cost of the replication strategy at:

$$\phi V^*(t) \equiv \phi V^F(t) - \mathbf{CVA} = \phi V(t) - \mathbf{FVA} - \mathbf{CVA} \tag{58}$$

[8] We are aware of that the pricing is much more involved as the one we are presenting, since each component of the price is affected by the others and a simple decomposition as the one we are presenting here is possible only by disregarding them. Anyway, it is also true that the error we make in doing this is not substantial.

As we have seen before for the replicator, the value of the contract depends on the side (long or short), on the funding costs (the **FVA**) and on the expected loss given the default of the counterparty (the **CVA**). Both quantities operate in the same way: they lower the value to the bank when it is long the contract, and they increase the negative value (i.e.: make it more negative) when it is short the contract. The two V^{Bid} bid and V^{Ask} ask values at which the bank can fairly trade the contract, expressing them in absolute terms, are:

$$V^{Bid} = V - \mathbf{FVA} - \mathbf{CVA} \leq V$$

and

$$V^{Ask} = V + \mathbf{FVA} + \mathbf{CVA} \geq V$$

which confirms the results in Castagna [5].

We can make the following considerations:

- in a multi-period setting, where existing debt cannot be renewed continuously, the production activity of derivative contracts by the bank, although theoretically risk-free,[9] does not imply a marginal funding cost equal to the risk-free rate, but on the contrary a positive funding spread, if the replication strategy prescribes to borrow cash;
- when internally evaluating a derivative contract, if the banks pays for whatever reason a funding spread, then the standard replication theory that assumes a unique lending and borrowing risk-free rate has to be accommodated to include also a funding valuation adjustment (**FVA**);
- a credit valuation adjustment (**CVA**) enters also in determining the value to the bank;
- eventually the default of the bank does not matter. This means that the value to the bank is determined as it were not subject to default, even if it is taken into account when starting the evaluation process through $\mathbf{PD}_1$ and $\mathbf{PD}_V$. As a consequence the debit value adjustment **DVA** is not present in the evaluation equation (58);
- the **DVA** is the **CVA** seen from the counterparty's point of view. When, in the bargaining process, the bank has to yield the **DVA** as a compensation to the counterparty, it represents a cost and it cannot be compensated by the **FVA**, as it is usually affirmed nowadays in many works. On the other end, the **FVA** and the **DVA** are not two ways to call the same quantity, if not in the particular case of a loan contract (see Castagna [6] for a discussion on this point).
- there is no double counting of **DVA** and **FVA**: they operate with different signs on the value of the contract. The bank should try not to pay the first one, since it

[9] Again, disregarding possible model risks.

cannot be replicated under realistic assumptions;[10] the second one can be included in the value of the contract and should be considered as a production cost.

Proposition 5.1. *When evaluating a derivative contract whose counterparty's default does not imply the bank's default, the correct and theoretically consistent value is given by the otherwise risk-free theoretically fair price plus the credit and funding valuation adjustments. The sign of the adjustments depend on the long or short position taken into the contract by the bank. The bank's default does not enter in the evaluation process, so that no debit valuation adjustment is considered.*

The Default of the Counterparty Implies the Default of the Bank

Assume now that derivative contract has a notional such that the default of the counterparty implies the default of the bank. The valuation cannot be operated by pretending that the contract is a standalone operation without any regard to the remaining parts of the balance sheet. We need to resort to numerical procedures, but at least formally we come up with a valuation formula.

By rewriting equation (51), we have:

$$\begin{aligned}\mathbf{VB}(t) =& \mathbf{E}\left[X_1(ms_1 - f_1) + \Pi(T) + E\right] \\ &= \left[X_1(ms_1 - f_1) + \overline{\Pi} + E\right](1 - \mathbf{PD}_1)(1 - \mathbf{PD}_V) + \epsilon \\ &= E\end{aligned} \tag{59}$$

where $\epsilon = \overline{R}_1^* \mathbf{PD}_1(1 - \mathbf{PD}_V) + \overline{R}_2^* \mathbf{PD}_V(1 - \mathbf{PD}_1) + \overline{R}_{1,2}^* \mathbf{PD}_1 \mathbf{PD}_V$. Let $\mathbf{PD}_{1,2} = 1 - (1 - \mathbf{PD}_1)(1 - \mathbf{PD}_V)$. As before, we solve for the fair value of V and we get:

$$\phi V^*(t) = \phi V(t) - \mathbf{FVA} - \frac{E - [E + X_1(ms_1 - f_1)(1 - \mathbf{PD}_1)(1 - \mathbf{PD}_V)] - \epsilon}{(1 - \mathbf{PD}_1)(1 - \mathbf{PD}_V)} \tag{60}$$

which is the value of the risk-free fair value plus the cash needed to compensate the other adjustments. It is worth stressing that the **CVA** has been replaced by another quantity that considers the limited liability of the bank's shareholders, along with the other assets included in the balance sheet, so as to determine the correct loss upon the counterparty's default. This loss is smaller than the **CVA**, which is the maximum loss that the shareholders may suffer when the default of the counterparty does not trigger the banks default and hence no limited liability can be claimed.

As in the case of a generic asset, we can define the **LLVA** for a derivative contract as:

$$\begin{aligned}\mathbf{LLVA} =& \left[\phi V(t) - \mathbf{FVA} - \frac{E - [E + X_1(ms_1 - f_1)(1 - \mathbf{PD}_1)(1 - \mathbf{PD}_V)] - \epsilon}{(1 - \mathbf{PD}_1)(1 - \mathbf{PD}_V)}\right] - \\ &[\phi V(t) - \mathbf{FVA} - \mathbf{CVA}] \\ &- \frac{E - [E + X_1(ms_1 - f_1)(1 - \mathbf{PD}_1)(1 - \mathbf{PD}_V)] - \epsilon}{(1 - \mathbf{PD}_1)(1 - \mathbf{PD}_V)} + \mathbf{CVA}\end{aligned} \tag{61}$$

[10] See Castagna [6] for a formal proof.

so that the total value of the derivative is:

$$\phi V^*(t) = \phi V(t) - \mathbf{FVA} - \mathbf{CVA} + \mathbf{LLVA} \tag{62}$$

Since the **LLVA** is a positive quantity, it plays an opposite role that the **CVA**, *i.e.*: it makes the bank pay more when entering a long a position, and require a smaller premium when entering a short position in the derivative contract.

It is likely useful to calculate the debit value adjustment that the bank can be reasonably expected to be requested by the counterparty, to compensate the bank's default risk it bears (the **DVA** from the bank's point of view is the **CVA** from the counterparty's point of view).

Assuming we are still in a market where perfect information is available to all agents, the expected recovery $\overline{\mathbf{Rec}}_V^C$ on the derivative contract that the counterparty calculates, is:

$$\overline{\mathbf{Rec}}_V^C = \sum_{j=1}^{J} \min\left[\frac{X_1 \mathbf{Rec}_{1_j} + \phi(\Delta S \mathbf{1}_{\{-\phi\Delta>0\}} + \beta^F B \mathbf{1}_{\{-\phi\beta^F>0\}}) + E}{X_1 + |\mathbf{ENE}| + |\Delta S \mathbf{1}_{\{-\phi\Delta<0\}}| + |\beta^F B \mathbf{1}_{\{-\phi\beta^F<0\}}|}; 1\right] p_{1_j} \tag{63}$$

Equation (63) is worth a few comments: firstly, the derivative contract is excluded from the calculations, since it is considered for its expected liability aspect (the **ENE**), so that it cannot be part of the (possibly residual) value of the assets upon the bank's default (this is why the recovery on the **EPE** never appears at the denominator). Secondly, if the bank went bust, that means that the issuer of the asset A_1 defaulted, so that only a fraction of the face value X_1 is recovered. Thirdly, the replicating portfolio can be an asset and/or a liability, depending on the signs of the quantity entering in it; this explains the indicator functions at the numerator of fraction which make the components of the replication portfolio an asset when their quantity is positive. If their quantity is negative, the absolute value adds to the total amount of the bank's liability, which is at the denominator of the fraction. The total residual assets are divided by total liability to have the average recovery that each of them will collect (we are not considering priority rules in the distribution of residual assets).

The **DVA** is then:

$$\mathbf{DVA} = (1 - \overline{\mathbf{Rec}}_V^C) \times \mathbf{ENE} \times \mathbf{PD}_B = \overline{\mathbf{Lgd}}_V^C \times \mathbf{ENE} \times \mathbf{PD}_B \tag{64}$$

which is the expected loss suffered by the counterparty on the expected exposure (**ENE**) when the bank defaults. The probability of the bank's default depends the probability of the counterparty's default $\mathbf{PD}_V$: this is a circular relationship that causes no problem when the contract is small compared to the total assets of the bank, so that we fall back in the first case examined. If the notional amount is big enough to produce the bank's default when the counterparty goes bankrupt, $\mathbf{PD}_B$ should be adjusted by excluding the effect of $\mathbf{PD}_V$. When the counterparty's default triggers the bank's default, the value of the contract is positive to the bank, since it suffers a loss of the **EPE** such that the total assets are not able to cover the outstanding liabilities. But this means that the cases when the bank defaults after the counterparty's default can never be applied to the **ENE**.

All this discussion on the **DVA** is to show that the claim usually stated in theoretical works (and, alas, often also in practice) are not very soundly grounded: the **DVA** is always a very difficult quantity to deal with and it may cause many inconsistencies when taken into account not in a proper fashion. As a general rule, the **DVA** is simply a cost for the bank corresponding to the counterparty's **CVA**. The **DVA** can be computed by the usual tools only when the amount of the deal is such that the counterparty's default does not trigger the bank's default. In any case we would like to stress the fact that **DVA** is quite a different quantity from the quantity **LLVA** defined before.

The following considerations are in order:

- when the default of the counterparty can trigger the default of the bank, then the evaluation of the derivative contracts must consider the entire balance sheet of the bank: the process is quite cumbersome;
- in very general terms, the value of the contract is made of the following components:
  1. the theoretical value of an otherwise identical risk-free contract;
  2. minus the **FVA**, computed as in the case examined before, depressing the value of a long position, or making more negative the value of a short position;
  3. the maximum loss given the default of both the issuer of asset A_1 and of the counterparty (and, hence, of the bank): this equals the equity capital E times the joint probability of default $\mathbf{PD}_{1,2}$. The maximum loss is limited to the equity due to the limited liability of the shareholders. This quantity can be roughly assimilated to the **CVA** component of the first case examined above and it operated in the same way on the initial value of the contract;
- The justification adduced by some authors (see Hull&White [12], [14] and [13], for example) that the **DVA** is a benefit upon default due to the limited liability of the shareholders, is partially confirmed only in the cases when the derivative contract has a big impact on the balance sheet, otherwise the **DVA** should be considered as a cost (if paid). Also the identification of the **FVA** with the **DVA** (or a part of it) is not justifiable.

Proposition 5.2. *When evaluating a derivative contract whose counterparty's default can trigger the bank's default, the correct and theoretically consistent value is given by the otherwise risk-free theoretically fair price plus the funding valuation adjustments, the maximum loss suffered on the joint default and the adjustment due to the possibility to cover losses with other non defaulting assets and the limited shareholders' liability in case of bank's default. The sign of the adjustments depend on the long or short position taken into the contract by the bank. The bank's default enters in the evaluation process only as far as the maximum loss is considered. The* **LLVA***, although playing a similar role, is only roughly ascribed to the* **DVA***.*

Balance Sheet Shrinkage, Funding Benefit and No Arbitrage Prices

Some authors (Burgard&Kjaer [2] and Morini&Prampolini [18], for example) recur to the "funding benefit" argument to justify the inclusion of the **DVA** into the valuation process of the derivative contracts by the bank. On the other hand, they see the **DVA** strictly related to the the cash received when entering in the derivative contracts, so that **DVA** clashes with the −**FVA**, in the sense that the former should be seen as a funding benefit.

For some types of derivative contracts, such as an uncollateralised short position in a plain vanilla put option, the quantity $-\mathbf{FVA}$ may be the same as the **DVA**. This happens when the underlying asset is repo-able, *i.e.*: can be bought via a repo transaction so that the actual financing cost is the implied repo rate, which is very near to the risk-free rate and the related cost of funding can be neglected. Hence, the only cost of funding refers to the cash the bank needs to pay the option's premium (in this case it is zero, since the premium is received): in Castagna [6] we identified this part of thetotal **FVA** as $\mathbf{FVA^P}$, to distinguish it from the other component of the total funding value adjustment, $\mathbf{FVA^U}$, due to the financing costs of the position in the underlying, if any.[11]

On the other hand, limiting the **FVA** to the only $\mathbf{FVA}^P$ component can be incorrect for all those contracts that can have a double sided value for either counterparties and that generally start with zero value for both, so that no premium is exchanged between parties. In this case the **FVA** refers to possible future cash-flows and it definitely differs from the **DVA**.[12]

We have seen above, confirming the results achieved in Castagna [4] and [5], that the **DVA** and the **FVA** operate with opposite signs on the value to the bank of the contract: in case the bank, when going long (short) the contract, is forced to pay the **DVA**, it makes the value higher (lower) than the level the bank would be willing to pay (receive) at the inception; at the same time, the acceptable value for the bank for a long (short) position is decreased (increased) by the **FVA**. If we accept the inclusion of $-\mathbf{FVA}$ into the valuation of derivative contracts, then there could be some overlapping of the **DVA** and the **FVA**, seen as a funding benefit and accounted for with the opposite sign.

It seems here that we have an opposite view on something upon which an agreement could (and should) be reached. The difference between our view and the one originated by the "funding benefit" argument lies in the balance sheet shrinkage: basically, this activity consists in the reduction of the outstanding debt as soon as a positive cash flow is received by the bank. In this way, both assets (the cash) and liabilities (debt) decrease and the bank pays less funding costs.[13]. We have shown in the two cited works that the "funding benefit" is a badly posed concept, and it would be better to talk about balance sheet shrinkage instead; moreover, the funding benefit can be used to replicate the **DVA** only under very specific (and very likely unrealistic) conditions.

When the valuation of a derivative contract relies on the balance sheet shrinkage policy, an immediate consequence is that any cash-flow, either positive or negative, is present valued by adding funding costs. The reason is simple: if the cash-flow is negative, the bank must finance it by borrowing money and pay at the expiry of the loan also the

[11]Hull&White [12], [14] define these two quantities as **DVA**1 and **DVA**2, respectively.

[12]See Castagna&Fede [9], chapter 12, for the case of uncollaterlised swap contracts.

[13]If the equity capital (E) is constant in time, balance sheet shrinkage is the same as the activity currently denoted as *deleveraging*. It should be noted, though, that the balance sheet *delevarage* can be alternatively achieved by only increasing the equity capital, thus simply decreasing the amount of debt capital as a percentage of the total liabilities

funding costs (the difference between the actual and the risk-free interests); if the cash-flow is positive, the bank can buy back some outstanding debt, thus partially saving the future expenditure of funding costs, which then add to the present value of the received cash.

The assumptions, implicit in the possibility to implement such balance sheet shrinkage policy, are that the bank has some outstanding debt to buy back, preferably with the same expiry as the derivative contract (a fairly realistic assumption) and that the buy back can be done relatively easily with low transaction costs (this could hardly be met in practice, since not all outstanding debt is liquid and dealing with tight bid/ask spreads in the market).

Assume the shrinkage policy is feasible and let us modify the evaluation process of contract by the bank by including it. In practice, if $-\phi\beta^F > 0$ we modify equation (51) as follows:

$$\begin{aligned}\mathbf{VB}(t) =& \mathbf{E}\left[X_1 ms_1 - (X_1 - |\phi\beta^F B|)(1+f_1) + \phi V(T) - \phi\Delta S + E\right]\\ &= \left[X_1 ms_1 - (X_1 - |\phi\beta^F B|)(1+f_1) + \phi V(T) - \phi\Delta\overline{S} + E\right](1-\mathbf{PD}_1)(1-\mathbf{PD}_V)\\ &+ \overline{R}_1^*\mathbf{PD}_1(1-\mathbf{PD}_V) + \overline{R}_2^*\mathbf{PD}_V(1-\mathbf{PD}_1)\\ &+ \overline{R}_{1,2}^*\mathbf{PD}_V\mathbf{PD}_1 = E\end{aligned} \tag{65}$$

where

$$\overline{R}_1^* = \sum_{j=1}^{J} \max\left[X_1\mathbf{Rec}_{1_j} - (X_1 - |\phi\beta^F B|)(1+f_1) + \phi V(T) - \phi\Delta\overline{S} + E; 0\right] p_{1_j},$$

$$\overline{R}_2^* = \sum_{l=1}^{L} \max\left[X_1 ms_1 - (X_1 - |\phi\beta^F B|)(1+f_1) + \mathbf{Rec}_{V_l}\mathbf{EPE}_V + \mathbf{ENE}_V - \phi\Delta\overline{S} + E; 0\right] p_{V_l}$$

and

$$\begin{aligned}\overline{R}_{1,2}^* =& \sum_{j=1}^{J}\sum_{l=1}^{L} \max\left[X_1\mathbf{Rec}_{1_j} - (X_1 - |\phi\beta^F B|)(1+f_1)\right.\\ &\left. + \mathbf{Rec}_{V_l}\mathbf{EPE}_V + \mathbf{ENE}_V - \phi\Delta\overline{S} + E; 0\right] p_{1_j} p_{V_l}.\end{aligned}$$

In equation (65) the positive cash-flows related to the long bank account position $-\phi\beta^F B > 0$, is deducted from the outstanding amount of debt $D_1 = X_1$. This will generate a saving equal to the smaller interests paid on the reduced notional of debt. To simplify things, we did not analysed how the creditors of the bank would update f_2 when including the balance sheet policy; moreover we did not consider the fact the the outstanding debt would probably deal in the market at a price lower from X_2, which is the level we implicitly assumed.

In the standard replication argument presented above, when $-\phi\beta^F > 0$ we have $\beta^F = \beta$ and $\mathbf{FVA} = 0$. If the positive cash-flows are not invested in a risk-free bank account, but they are used to shrink the balance sheet, they will implicitly yield f_1 instead of the risk-free rate (set equal to zero in this analysis). If we limit the analysis to the case when the notional of the derivative contract is small compared to the total assets, we set

$f_2 = f_1$, and we repeat the same reasoning as that one presented above, then it is easy to realise that the value of the contract to the bank will be:

$$\phi V^*(t) \equiv \phi V^F(t) - \mathbf{CVA} = \phi V(t) - \phi\mathbf{FVA} - \mathbf{CVA} \tag{66}$$

If the bank is long, the value will be lower, as in the previous case when the shrinkage policy is not adopted; when the bank is short, the value of the contract will be less negative to the bank, so that it will be ready to accept a lower premium from the counterparty to enter in the short position of the contract. In contracts where the value is always positive to one of the two parties, when the bank is long it will accept to pay the **FVA** but not the **DVA**; when it is short, it will yield the **DVA** to the counterparty, thus accepting a lower premium. But this is theoretically the same quantity as the (negative) of the **FVA**[14] considering the balance sheet shrinkage policy, or the funding benefit as it is generally mentioned. The problem related to the double counting of the **DVA** and the **FVA** stems from this equivalence.

The balance sheet shrinkage policy seems an effective way for the bank to be aggressive in offering derivative contracts to the counterparties: it justifies the inclusion of the **DVA**, seen as a funding benefit, or $-$**FVA**. Although the two quantities may not fully coincide in any case, they are very similar in most of cases.

Nonetheless, this policy has many drawbacks that make it less enticing than it may appear at first look. We will briefly list some related problems:

- as mentioned above, it is not always possible to find suitable bonds (outstanding debt) to buy back in the market at fair prices;
- the policy produces counter-intuitive results, pushing the bank to progressively willingly lower the selling price of derivative contract when its creditworthiness worsens. It can be forced to do so because the counterparty can have enough bargaining power to ask for the inclusion of the **DVA** in the traded price, but this has to be seen as a cost and not an adjustment to lightheartedly accept;
- the shrinkage is possible only when there is a premium that must be paid by the counterparty (the asset can be financed in most of cases by a repo transaction, although this is not always feasible). In contracts such as uncollateralised forwards and swaps the bank still will pay funding costs related to the replication strategy and possibly also a **DVA**, but they act with opposite signs, as we have shown before;
- the funding benefit argument is basically a claim on the replicability of the **DVA** more than the justification of the inclusion of the **FVA** with reverse sign. The **DVA** can be replicated only under very restrictive conditions.

Proposition 5.3. *If the balance sheet shrinkage policy is included in the evaluation a derivative contract,the correct and theoretically consistent value is given by the otherwise risk-free theoretically fair price plus the credit and funding (cost) valuation adjustments*

[14] In practice there are some differences between the **FVA** and the **DVA**, due to the fact the different recoveries are attached to the outstanding debt and the exposure of the derivative contract.

and the debit value adjustment (or alternatively the funding (benefit) value adjustment). The sign of the adjustments depend on the long or short position taken into the contract by the bank.

We already supported elsewhere[15] the idea that the balance sheet shrinkage should not be considered a sound assumption when evaluating derivative contracts, which means that the **DVA** is treated always as a cost (if paid) and that the **FVA** never enters in the internal value of the contract with the negative sign (*i.e.*: as a funding benefit). This means that the internal value is the one we have defined in equation (58), when no balance sheet shrinkage was considered.

Example 5.1. *We sketch here the effects of the assumptions made by the bank regarding the replication strategy (i.e.: feasible or not feasible balance sheet shrinkage) when a derivative contract is closed.*

To make things concrete, consider an option V on an asset S, expiring in T. Assume also that going short and long the underlying asset S can be done with repo/reverse-repo transactions, paying a repo rate (approximately) equal to the risk-free rate. In this case the funding is due only to the payment of the premium. The bank evaluates the "production" costs to sell it (i.e.: enter in a short position) and, applying equation (58)*, it gets:*

- $-V(0) = -10$, *value of the risk-free option evaluated with the chosen model (e.g.: B&S);*
- $-$**FVA** $= 0$, *since the premium is received;*
- $-$**CVA** $= 0$, *since no counterparty risk exits for a short position in an option;*
- $V^*(0) = -10$

The bank should sell the option at a price $P = 10$: in the is case the P&L $= P - V^* = 0$ at the inception.

The counterparty of the bank in this contract is another bank with an equal bargaining power, so that it manages to charge **DVA** *in the option price, since it wants to be remunerated for the counterparty risk related to the bank's default probability (the* **DVA** *is the* **CVA** *from its standpoint). Given the bank's default* $\mathbf{PD}_B$ *and the loss given default of the option, the counterparty calculates the* **DVA** $= 1$*. So the traded price of the contract is:*

- $-V(0) = 10$
- $+$**DVA** $= 1$
- $-$**FVA** $= 0$
- $P = -10 + 1 = 9$

[15] See Castagna [5].

The bank sells the option at the price $P = 9$, so that the $P\&L = P - V^* = -1$ at the inception, given the value the bank attached to this contract.

Consider now the case when the bank assumes that it is able to operate the balance shrinkage as an ordinary and routine liquidity policy. The option "production" cost (i.e.: the value to the bank) is in this case derived by equation (66):

- $-V(0) = -10$
- $+$**FVA** $= 1$, *the funding benefit, originated by own bonds' buy-back;*
- $-$**CVA** $= 0$, *since no counterparty risk exits for a short position in an option;*
- $V^*(0) = -10 + 1 = -9$

The bank is happy to sell the option at $P = 9$ and the $P\&L$ in the trade above would be $P - V^* = 0$ at the inception. Holding the $P\&L = 0$ up to the expiry (or, the end of the replication) depends on the ability to actually and effectively implement the balance sheet shrinkage. In practice, even if the counterparty has not a great bargaining power, it will be compensated for the **DVA** *by the bank all the same, since it includes the funding benefit in its value. In the first case (no balance sheet shrinkage), during the bargaining process, the bank will reluctantly yield the* **DVA** *because of the counterparty's strength; in the second case (balance sheet shrinkage) the bank will include the* **DVA** *(seen as a funding benefit fva) even if not requested by the counterparty.*

Consider now the case when the bank goes long the option. In this case the counterparty risk is relevant; the bank measures the risk and sets **CVA** $= 0.75$. *The option "production" cost (value to the bank) is given, again, by equation (58):*

- $V(0) = 10$
- $-$**FVA** $= -1$ *since the bank pays the option's premium, which needs to be funded;*
- $-$**CVA** $= -0.75$, *the compensation for the counterapry risk;*
- $V^*(0) = 10 - 1 - 0.75 = 8.25$

The bank should buy the option at a price $P = 8.25$, so that $P\&L = V^* - P = 0$ at the inception.

Let us assume that the counterparty is a bank with a great bargaining power, so that it does not accept to pay the **CVA** *(i.e.: to compensate the bank for the counterparty risk) and the* **FVA** *(i.e.: the costs the bank pays to fund the premium payment above a risk-free agent). The price of the contract the counterparty manages to receive, is:*

- $V(0) = 10$
- $-$**CVA** $= 0$
- $-$**FVA** $= 0$
- $P = 10$

The bank buys the option at $P = 10$, the so it marks a loss since $P\&L = V^* - P = -1.75$ at the inception.

If the bank thinks that the balance sheet policy can be hardly implemented in practice, the sell and buy prices at which it is willing to trade, are

$$V^{Bid} = 8.25 < V(0) = 10 < V^{Ask} = 10.$$

If the bank evaluates contracts under the assumption that the of the balance sheet shrinkage is a feasible liquidity policy, regularly and effectively run in reality, then

$$V^{Bid} = 8.25 < V^{Ask} = 9 < V(0) = 10.$$

In this case the bank is willing to sell the option at a price which is lower than the otherwise risk-free value, relying on the funding benefit it receives. This point may also raise some eye brows, besides the considerations on the feasibility of the balance sheet shrinkage policy.

6 Conclusion

We have proved that the classical results on the evaluation of investments by Modiglian&Miller [17] and Merton [16] hold only when the bank holds a single asset: in this case the value of the investment is independent from the capital mix (*i.e.*: the amount of debt and equity capital used to fund it). In a multi-period setting, for an additional investment the results do not hold (or they hold partially), so that the way it is financed does matter and the funding cost enter in the evaluation process. The main point in this framework is that the existing debt cannot be updated, until its expiry, to reflect the new risk of the bank's total assets, so the total financing cost is higher than the level that would prevail if all the debt could be freely renewed. Moreover, in the real world the equity is used to cover losses generated indistinctly by all assets: the limited shareholders' liability produces its effects on the evaluation process only if the held amount of the asset is large enough to trigger the bank's default when the issuer goes bankrupt. This result gives a theoretical support to the well established practice in the financial industry to take into account funding costs and to charge them in the value of the asset when the bank's bargaining power allows to do so.

For derivative contracts the results we derived are similar: the counterparty's credit risk and the funding costs have to be included in the value of the contract to the bank, whereas the bank's own default does not play any role, thus the debit value adjustment should be excluded. When the bank embeds the balance sheet shrinkage policy in the evaluation process, then the negative of the funding cost, meant to be a funding benefit, is included in the value, producing the same (often identical) effect of the inclusion of the **DVA**. Since in our framework the **DVA** never enters in the value, the possible inclusion of the funding benefit will never generate any double counting effects. Anyhow, we think that relying on the balance sheet policy is rather incautious, since we think that in reality it cannot be implemented on the base required by the derivative market-making activity.

It is maybe interesting to note that the conclusion we have reached is the same as in Hull&White [12], [14] and [13], although partially and for a different reason. In fact, the two authors exclude the **FVA** in any case, either it enters in the value as a cost or as a benefit, mainly relying on the classical investment evaluation theory's result of

Modigliani&Miller. Besides, they show in [13] the possibility of arbitrages when the funding benefit $-\mathbf{FVA}$ is accounted for in the value of the contract. In any case, their aim is to support the theoretical result of one price dealing in the market.

In our framework, the **FVA** increases the value of short positions (makes it more negative) and decreases the value of long positions. So the **FVA** is always a positive or nil value. When the balance sheet shrinking is considered as a feasible and ever implementable policy, then the **FVA** can also enter with a negative sign (as a funding benefit) thus decreasing the value of both short and long positions.

We believe that, although theoretically possible, the balance sheet shrinkage policy should not be considered when evaluating derivative contracts. Firstly, the policy can be hard to be followed in practice and secondly it is not feasible when the contract is two sided (such as swaps) and it is dealt with no cash exchanged between parties. If and when it is actually implemented in practice, then a windfall for the bank occurs. If the buy-back of own bonds is not operated strictly following the rule, the bank should evaluate derivative contracts as we have indicated in formula (58), hence agreeing with H&W at least as far as the funding benefit is concerned. The funding benefit, if any, should be referred to the smoothing of the future cash-flows' profile of the derivatives' book, but this will be the object of future research.

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