Market Instruments for Collateral Management*

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Abstract

We analyse the most common market instruments to manage and optimise collateral allocation: the repo, the sell/buy back and security lending.

1 Introduction

In a previous article (Castagna [5]) we introduced a conceptual framework to analyse the supply and the demand of collateral internally originated by the banking activity; we focussed on the tools to manage both and the targets that should be aimed at by setting up effective processes to minimise costs and to monitor risk exposures.

In the present work, we ideally continue our study of the modern collateral management by analysing three types of contracts that are the basic market instruments to manage and optimise collateral allocation: the repurchase agreement, the sell/buy back and the security lending. For each of them, most updated legal frameworks and best practices will be thoroughly investigated. Moreover, attention will be given to their evaluation in terms of fair value and additional adjustments due to credit, liquidity and funding factors.

The three instruments are the building blocks for more complex contracts and sophisticated trading strategies aiming at an effective, enterprise-wide, collateral management.

2 The Repurchase Agreement (Repo)

Definition 2.1. *A* ***repurchase agreement****, commonly called* ***repo****, is a transaction in which one party agrees to sell security to another against the transfer of cash; at the same moment, the party also agrees to repurchase the same (or equivalent) security at a*

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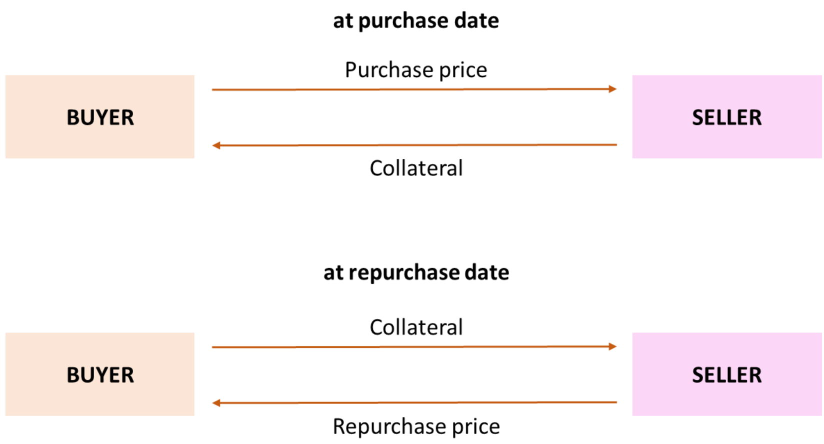

Figure 1: Obligations of a repo agreement.

specific price at a future date, when the contract will expiry. The party receiving security is also referred to as the ***buyer****; the party lending security is referred to as* ***seller****. When the transaction is seen as a secured cash loan, the repo buyer is also referred to as the* ***lender****, which receives security as collateral against a possible cash* ***borrower****'s default.*

Counterparties typically involved in a repo transaction are financial institutions (banks): although repos can be traded also between other types of economic agents (*e.g.*: corporate companies), in this paper we will assume that they always are banks. Figure 1 shows how the obligations of a repo agreement are fulfilled by the two parties.

In a repo transaction, there is an interest rate component which is implicit in the terminal price at which the repurchase occurs. At inception, the security are bought at the current market price plus the accrued interest to date; at the expiry, the security are re-sold at a predefined price equal to the original sale price (market price + accrued interest), plus an interest rate (the repo rate) agreed upon by the parties.

Repo can be security-driven or cash-driven transactions, depending on which is the primary interest of the two parties. If one of them is dealing to borrow the security (a security-driven transaction), the repo rate is typically set a level lower than current money market rates to compensate the lender of security. It is true that the repurchase price is higher than the initial selling price, because it will include the repo interests; nonetheless, the cash received by the seller can be reinvested for the duration of the contract in the money market, and the interest earned in this investment will be higher than the implied rate paid in the repo transaction, the difference being the compensation for the seller.

When one the two parties deals to receive cash against delivering security (a cash-driven transaction), the repurchase price is set so that the lender of cash (borrower of security) earns the equivalent of money market secured funding rate. The collateral represented by security protects the cash lender against the cash borrowers' default, so the repo rate should trade at an implicit rate lower than an unsecured funding rate, such as a money market deposit rate. We will return on this point later on.

In cash-driven repo transactions, a margin is often provided to the lender of cash by pricing the collateral security at the market price minus a "haircut", or an "initial margin"(we will discuss the difference between the two below). Therefore, the initial price is lower than the market price of the security. On the other hand, in security-driven deals, the lender of security will typically receive a margin by pricing security higher than their market value. The mechanics is similar in the two cases anyway.[1]

Repo agreements are quoted in terms of the repo rate. The repo rate is generally quoted on the basis of the day count/annual basis convention prevailing in the wholesale money market in the currency of the Purchase Price (*i.e.*:, the deposit and forward rate market).[2] In the Global Master Repurchase Agreement (GMRA, the standard general legal framework used by most banks), the repo rate is called the Pricing Rate.

A buyer in a repo may also allow the seller to replace some or all collateral securities during the duration of the repo. In practice, the seller, at any time between the purchase date and repurchase date, has the right to call for the buyer to return equivalent collateral securities in exchange for substitute collateral. For this right, the seller typically agrees to pay a higher repo rate.

As seen above, since the repo can be security-driven, the repo rate can be negative: this may happen when a particular collateral asset is in large borrowing demand and/or in scarce supply, so that it is said to go *on special*.

During the life of the contract, the collateral is legally property of the buyer, which means that all payments produced by the security are paid by the issuer directly to the buyer. However, under an economic point of view, the buyer can be considered just as the possessor, not the owner, of the security, because the repurchase price is fixed and there is the seller's obligation to repurchase the security. Hence both the risk and return on the collateral asset are kept by the seller.[3]

The guiding principle is that the seller should receive all income payments due on collateral asset as if the asset had not been repoed out. This means that the terms of a repo contract provides for the obligation for the buyer to make an equivalent and immediate[4] income payment to the seller (called "manufactured payment") every time the collateral asset produce a payment. Following the same criterion, if the issuer for some reason fails to make an income payment due on the collateral security, the buyer does not have to make the equivalent payment to the seller.

The transfer of legal title to collateral in the initial sale means that the buyer has the right to re-use, or re-hypothecate, the collateral during the duration of the repo. The

[1]Security driven repo transactions are very similar to security lending transactions when collateral is provided by the security borrower: the legal and economic differences will be clear after analysing the security lending later on.

[2]The two conventions prevailing in practice are actual/365 days and actual/360 days.

[3]Castagna and Fede [7] consider just the economic perspective when they analyse how the market instruments we study in this paper affect the amount of securities available for collateral (or, more generally, liquidity) purposes.

[4]On the same day as the corresponding income payment by the issuer.

right exits independently from the default of the seller. So the buyer can repo again the bought asset or sell it to a third party at any time: it only has to be able to return to the seller an equivalent amount of securities on the repurchase date. The right to re-use the bought asset is the reason why repo transactions are market instruments to manage and transform collateral and employ them in collateral strategies, as discussed in Castagna [5].

2.1 Quotation and Trading

When repo contracts are cash-driven, then they are typically quoted in terms of (repo) rate, given a standard haircut to be applied on the collateral asset: the collateral asset can be any from a pool of eligible assets that the market considers of suitable credit quality and liquidity. The haircut for these collateral assets is the same, since they have similar credit and liquidity characteristics. For these reasons, cash-driven repos are defined *General Collateral* (GC) and quotes are easily available for them in the market for a range of standard maturities running from O/N to 1 year.

If the repo is security-driven, then the quotation has to be defined between the counterparties, both in terms of repo rate and haircut to be applied on the asset: the stronger interest in borrowing the security, rather than the cash, may shift the repo rate to levels lower than the GC repo rate. Moreover, the haircut depends on the credit and liquidity characteristics of the collateral asset and it can be also quite different from the GC haircut (see also below for how the fair haircut is calculated). As mentioned before, the collateral asset in this repo are said to go *on special*, and repos are Specific Collateral (SC). Quotations are in terms of special repo rates and haircuts to be agreed by the parties.

Either the repo is GC or SC, once the repo rate and the haircut are agreed upon, the trading of the transaction involves the calculation of the Purchase Price and Repurchase Price: both can be easily calculated as follows. Consider a repo starting at time t and expiring in T and let $\mathbf{MV}(t)$ be the market value of the collateral asset: in case it is a bond, it is the dirty price (the quoted market clean price plus accrued interests from the last coupon payment date to date t); if it is (less commonly) an equity, it is just the market price.

The Purchase Price **PP** is the $\mathbf{MV}(t)$ deducted the haircut $\mathcal{H}(t)$, or the initial margin $\mathbf{IM}$(t) (see below for the more details):

$$\mathbf{PP} = \mathbf{MV}(t)(1 - \mathcal{H}(t)) \tag{1}$$

or

$$\mathbf{PP} = \frac{\mathbf{MV}(t)}{\mathbf{IM}(t)} \tag{2}$$

Let the repo rate of the transaction be r^E, then the Repurchase Price **RP** is computed as:

$$\mathbf{RP} = \mathbf{PP} \times [1 + r^E \times \tau^E(t,T)] \tag{3}$$

where $\tau^E(t,T)$ is the year fraction of the period $[t,T]$ calculated according to the day count and basis convention of the contract.

Example 2.1. *We present a practical example of how the Purchase and Reurchase Prices are computed for a repo traded on the 6^{th} August, 2014. The collateral asset is a German government bond, the Bund with annual fixed coupon with details shown in Table 1*

Face Value	100		
Coupon	3	Ref. Date	06/08/2014
Frequency	yearly	Clean Price	115.05
Last Payment	04/07/2014	Accrual Days	33
DC Conv.	Act/Act	Accrued Interests	0.2712
Expiry	04/07/2020	Dirty Price	115.32

Table 1: Details for the bond Bund 3% Jul 20, denominated in Euro.

In Table 2 we show the Purchase Price in the two cases when the haircut or the initial margin is used, given the market price and the accrual conventions to compute the market value of the asset **MV** *(the bond's dirty price).*

Haircut			Initial Margin		
Dirty Price		115.3212	Price		115.3212
Haircut	3%	- 3.4596	Initial margin	3%	- 3.3589
Purchase Price		111.8616	Purchase Price		111.9624

Table 2: Purcahse price when collateral is subject to haircut or initial margin.

Assume the repo has an expiry in three months, or 92 days from the trading date, and that the agree repo rate $r^E = 1.75\%$*; the notional amount is* $1,000,000.00$ *Euros. Given the money market convention for accrued interests of Act/360 (i.e.:* $\tau^E(0, 92dd) = 92/360$*, the initial cash paid by the buyer (lender) and the cash returned by the seller (borrower) are given in Table 3. The Repurchase Price is clearly the cash returned for 1 eur of notional amount.*

	Haircut	Initial Margin
Repo Cash	1,118,615.96	1,119,623.62
Bond Notional	1,000,000.00	1,000,000.00
Repo Rate	1.75%	1.75%
Maturity	06/11/2014	06/11/2014
Days	92	92
Repo Interests	5,002.70	5,007.21
Cash Returned	1,123,618.66	1,124,630.83
Repurchase Price	112.3619	112.4630

Table 3: Details of the repo contract when collateral is subject to haircut or initial margin.

2.2 Fair Haircuts and Initial Margins

As mentioned above, there are two alternative ways to adjust the value of the collateral asset sold in the repo to protect the buyer against the loss it may suffer when the seller defaults. the initial margin. The haircut and initial margin are typically applied in cash-driven transaction, where the collateral asset is the protection of the lender against the borrower default.

The initial margin is defined as:

$$\mathbf{IM}(t) = \frac{\mathbf{MV}(t)}{\mathbf{PP}} \tag{4}$$

where $\mathbf{IM}(t)$ is the initial margin at time t, $\mathbf{MV}(t)$ is the market value of the collateral and $\mathbf{PP}$ is the purchase price. The initial margin is expressed as a percentage on the market value: 100% means that no margin has been applied.

The second adjustment is the haircut $\mathcal{H}(t)$, defined as:

$$\mathcal{H}(t) = \frac{\mathbf{MV}(t) - \mathbf{PP}}{\mathbf{PP}} \tag{5}$$

where the notation is the same as above. The haircut is the percentage difference between the market value of the collateral asset and the purchase price.

For practical purposes, in the following, we may limit our analysis to the case when a haircut is applied: if an initial margin is applied instead, it is straightforward convert it to a haircut through the relation:

$$\mathbf{IM}(t) = \frac{1}{1 - \mathcal{H}(t)} \tag{6}$$

The problem the seller (or, the cash lender) faces is how to set a fair level of the haircut. Assume the seller is a bank and that it enters a repo transaction at time t, expiring in T, with a counterparty, indicated with d, which may go bust before the end of the contract.

We will present an approach to determine the fair haircut that is derived from a model outlined in chapter 8 of Castagna and Fede [7], although we here substantially modify it. In any case, we retain the main feature of the model allowing for a wrong-way risk between the default of the counterparty and the default of the issuer of the collateral security, in this case of the bond.

The wrong-way risk is very common in repo transactions because often the collateral bonds are issued by the government of the country of the cash borrower. Typically a strong correlation exists between the default probabilities of the government bonds and the banks, for the simple fact that banks have on their balance sheet huge quantities of such bonds, if only to build up liquidity buffers. When there are crisis on the sovereign debt, the **PD** of the banks of the countries under pressure increases as a result of the higher probability of a default of the government whose bonds are held by the bank.

To model the default of the counterparty (borrower) d, we introduce an indicator function $D_d(0, T)$ equal to 1 if the default of the counterparty occurred between times t and T, and it is equal to 0 otherwise. We take the approximation

$$D_d(0, T) = D_d^I(t, T) + \xi_d D_d^C(t, T)$$

where $D_d^I(t, T)$ is the indicator of the event that the idiosyncratic default process for d has jumped by T[5], whereas $D_D^C(t, T)$ is the indicator for the event that the common default

[5]We consider the default a jump process, accordingly with a reduced-form default modelling we are implicitly adopting.

process has jumped by T; finally ξ_i is the indicator of the event that d goes bust at the first common credit event.[6] In the framework we introduced above $\mathbf{Pr}(\xi_d = 1) = p_d$, where p_d is the factor linking the intensity of the common default event to the total intensity of the counterparty d, and in t (given that the counterparty did not default before):

$$\mathbf{PD}_d^I(t,T) = \mathbf{Pr}[D_d^I(t,T) = 1]$$

$$\mathbf{PD}_d^C(t,T) = \mathbf{Pr}[D_d^C(r,T) = 1]$$

Assume the collateral asset is a bond expiring in T_B: the dirty price is $\mathbf{B}(t,T_B)$. The cash transferred by the lender (buyer) to the borrower (seller) at inception is the same as the value of the bond deducted the fair haircut. Let $V_{\mathbf{Coll}}(t) = \mathbf{MV}(t)$ be the value of the bond used as collateral in t:

$$V_{\mathbf{Coll}}(t) = N_{\mathbf{Coll}}\mathbf{B}(t,T_B).$$

where $N_{\mathbf{Coll}}$ is the notional of the collateral bond. In a repo transaction, the lender seeks to make the expected loss, given the borrower's default, equal to zero over a given period from t to T. Assuming the default is observed at the expiry T of the contract (although it may occur at any time between t and T), the expected loss (**EL**) is equal to the expected exposure at default (**EAD**, assumed to be fully lost) minus the value of the collateral, in the even of a default of the counterparty d. The **EAD** is simply the amount lent (assuming no scheduled repayments between 0 and T) plus the interests, so that:

$$\mathbf{EL}(t,T) = \mathbf{E}\Big[\max[\mathbf{EAD} - V_{\mathbf{Coll}}(T), 0]\big|D_d(t,T) = 1\Big] \tag{7}$$

The idea we follow is that the lender, in setting the fair haircut, seeks to set at 0 the **EL** considering the minimum value attainable by the collateral bond at a given confidence level, say 99%. Let $v_{\mathbf{Coll}}(T)$ be this minimum value, corresponding to the maximum expected loss **EL** at the chosen confidence level; we rewrite (7) as:

$$\mathbf{EL}(t,T) = \mathbf{E}\Big[\mathbf{EAD} - v_{\mathbf{Coll}}(T)\big|D_d(t,T) = 1\Big] \tag{8}$$

The expected exposure is $\mathbf{E}[\mathbf{EAD}|D_d(t,T) = 1] = V_{\mathbf{Coll}}(t)(1 - \mathcal{H}(t))[1 + r^E \times \tau(t,T)]$, while the expected (minimum) value of the collateral is:

$$\begin{aligned}&\mathbf{E}[v_{\mathbf{Coll}}(T)|D_d(t,T) = 1] = \\ &\mathbf{E}[v_{\mathbf{Coll}}(T))(1 - D_{\mathbf{Coll}}(t,T)) + (v_{\mathbf{Coll}}(T) - \mathbf{Lgd}))D_{\mathbf{Coll}}(t,T)\big|D_d(t,T) = 1]\end{aligned}$$

where $D_{\mathbf{Coll}}(t,T)$ is the indicator function equal to 1 if the issuer of the collateral bond has gone bankrupt by time T, and **Lgd** is the loss generated by the default event and it

[6] This approximation has been used by Duffie and Pan [9]. We ignore the double-counting of defaults that occurs from both common and idiosyncratic credit events. The approximation also under-counts defaults associated with multiple common credit events before time T. These two effects are partially offsetting each other.

is a percentage $l\%$ of the notional $N_{\mathbf{Coll}}$. The issuer's default is modelled similarly to the counterparty's default, so we have:

$$D_{\mathbf{Coll}}(t,T) = D^{I}_{\mathbf{Coll}}(t,T) + \xi_{\mathbf{Coll}} D^{C}_{\mathbf{Coll}}(t,T)$$

and $\mathbf{Pr}(\xi_{\mathbf{Coll}} = 1) = p_{\mathbf{Coll}}$, $\mathbf{PD}^{I}_{\mathbf{Coll}}(t,T) = \mathbf{Pr}[D^{I}_{\mathbf{Coll}}(t,T) = 1]$, and $\mathbf{PD}^{C}_{\mathbf{Coll}}(t,T) = \mathbf{Pr}[D^{C}_{\mathbf{Coll}}(r,T) = 1]$. It is immediate to see that the parameters p_d and $p_{\mathbf{Coll}}$ play the role of the "correlation"[7] between the default of the borrower and of the bond's issuer, since the higher they are (everything else being equal), the more likely the default is triggered by the common event $D^{C}_{\mathbf{Coll}}(t,T)$, thus having two simultaneous defaults. Define:

$$P_1 = (1 - \mathbf{PD}^{C}(t,T))\mathbf{PD}^{I}_{\mathbf{Coll}}(t,T)$$

and

$$P_2 = \mathbf{PD}^{C}(t,T)((1 - p_{\mathbf{Coll}})\mathbf{PD}^{I}_{\mathbf{Coll}}(t,T) + p_{\mathbf{Coll}})$$

Equation (8) can be more explicitly written as::

$$\mathbf{EL}(t,T) = \mathbf{EAD} - v_{\mathbf{Coll}}(T) + \left[[P_1 \mathbf{PD}^{I}_{d}(t,T) + P_2((1-p_d)\mathbf{PD}^{I}_{d}(t,T) + p_d)]\mathbf{Lgd} \right] / \mathbf{PD}_d(t;T) \tag{9}$$

On the right hand side, we have the exposure at default minus the minimum value of the collateral at the expiry. The amount in the square brackets is the expected value of the loss on the bond in the event of the counterparty's default, given that also the collateral bond issuer's default event is triggered: we will denote this amount in what follows as $\mathbf{E}[\mathbf{Lgd}\, D_{\mathbf{Coll}}(t,T) | D_d(t,T) = 1]$. The higher the volatility of the price of the collateral bond, the lower the quantity $v_{\mathbf{Coll}}(T)$ will be. Actually, it is quite sensible to assume that a bond with a longer maturity (and hence duration) would be subject to a higher haircut than shorter maturity bond, due to the greater risk that, in case of default of the borrower, the collateral has a lower minimum market price, at the chosen confidence level.

The fair haircut to apply to the value of the portfolio of bonds at the inception of the repo is the level of $\mathcal{H}(t)$ that makes $\mathbf{EL} = 0$. In formula:

$$(1 - \mathcal{H}(t))V_{\mathbf{Coll}}(t)[1 + r^E \times \tau(t,T)] = v_{\mathbf{Coll}}(T) - \mathbf{E}[\mathbf{Lgd}\, D_{\mathbf{Coll}}(t,T) | D_d(t,T) = 1] \tag{10}$$

or

$$\mathcal{H}(t) = \frac{V_{\mathbf{Coll}}(t)[1 + r^E \times \tau(t,T)] - v_{\mathbf{Coll}}(T) + \mathbf{E}[\mathbf{Lgd}\, D_{\mathbf{Coll}}(t,T) | D_d(t,T) = 1]}{V_{\mathbf{Coll}}(t)[1 + r^E \times \tau(t,T)]} \tag{11}$$

It is interesting to measure how much the wrong-way risk impact on the haircut. To this hand we have to apply the formalae above assuming that no correlation exists between the counterparty and the issuer of the bond. In our set-up this means that the parameter

[7] We are aware that "correlation" is in this case, and later on, used in a loose meaning. Nonetheless, the effect produced by the parameters p_d and $p_{\mathbf{Coll}}$ is similar to that of a rigorously defined correlation between the defaults of the borrower and the bond's issuer, hence the use of the term.

p is equal to zero for each collateral asset's issuer and the borrower ($p_{\mathbf{Coll}} = p_d = 0$), or that the probability of a common credit event $\mathbf{PD}^C = 0$. Nonetheless, the total $\mathbf{PD}$ of both the issuer and the debtor are the same as the ones produced by the combined effect of the idiosyncratic and common factors. t Equation (2.2) modifies as follows:

$$\mathbf{EL}(t,T) = \mathbf{EAD} - v_{\mathbf{Coll}}(T) + \mathbf{PD}_{\mathbf{Coll}}(t,T)\mathbf{Lgd}$$

In practice, the haircut depends only on the default probability of the issuer of the collateral asset when there is no correlation between its default and the borrower's default. When the wrong way exists, that is: when the correlation between the two defaults is greater than zero, then the haircut depends also on the borrower's default and the level can be strongly affected. We will show this in an example.

Example 2.2. *We present a practical application of the framework to set fair haircuts outlined in the main text. We assume that the maturity of the repo contract is in one year and that the borrower and the issuer of the collateral bond have a default probability, respectively* $\mathbf{PD}_d$ *and* $\mathbf{PD}_{\mathbf{Coll}}$*, shown in Table 4. The same table also shows the value of the probability of occurrence of the common default event* $\mathbf{PD}^C$*.*

$\mathbf{PD}^C$	$\mathbf{PD}_d$	$\mathbf{PD}_{\mathbf{Coll}}$
1.00%	1.50%	1.40%

Table 4: Default probabilities of the borrower and of the issuer of the collateral bond.

Given this initial setting of the default values, we would like to investigate which is the fair haircut to apply on the bond, considering different correlations between the defaults of the borrower and of the bond's issuer. To this end, we present in Table 5 three sets of parameters $\mathbf{PD}^I_d$*,* p_d*,* $\mathbf{PD}^I_{\mathbf{Coll}}$ *and* $p_{\mathbf{Coll}}$ *that reproduce a medium, high and zero correlation. All the three sets return the default probabilities of Table 4, so the only difference is due to the probability of a joint default.*

$\mathbf{PD}^I_d$	p_d	$\mathbf{PD}^I_{\mathbf{Coll}}$	$p_{\mathbf{Coll}}$
1.00%	0.5	1.00%	0.4
0.50%	1.0	0.50%	0.9
1.50%	0.0	1.40%	0.0

Table 5: Parameters of the model to reproduce the three cases of medium, high and zero correlation between the default events of the borrower and of the issuer of the collateral bond.

In Table 6 we show the input data referring to the collateral bond. We consider two cases for the collateral bond maturity: a 2 year and a 10 year expiry. The different maturity will affect the minimum value $v_{\mathbf{Coll}}(1)$ *according to the volatility of the market prices, which we use in the calculations to compute the expected loss in formula* (2.2)*.*

Finally, in Table 7, we show the far haircuts to the price of the bond in the case its maturity is 2 or 10 years; the haircuts are calculated considering that the repo rate $r^E = 0$*. The three possible configurations of default correlation are reported. The results*

	Bond Expiry	
	2 Y	10 Y
N_{coll}	1	1
$\mathbf{B}(t,T)$	98.00	98.00
$v_{\mathbf{Coll}}(T)$	93.53	87.13
Price Volatility	2.0%	5.0%
Lgd	60	60

Table 6: Input data to set the fair haircut for the collateral bond.

make clear that the default correlation (wrong way risk) have a big impact on the value of the haircut; also considering the possible adverse price movements affects considerably the haircut, making it greater for the longer maturity bond.

	Bond Expiry	
Default Correlation	2Y	10Y
Medium	11.27%	14.00%
High	27.54%	28.46%
Zero	5.37%	9.57%

Table 7: Fair haircuts when the collateral bonds expire in 2 and 10 years, in the three default correlation cases.

The analysis above hinges on the simplifying assumption that the repo borrower's and issuer's default occur only at the expiry of the contract: this assumption can be reasonable for short term contracts, but for longer maturities it is too restrictive. To complete the analysis of the haircut modelling, we need to consider the more general case when the default of the borrower and of the issuer of the collateral bond may occur not only at the expiry of the repo contract, but at any time before the expiry. In this case it is more convenient, also for evaluation purposes, to deal with the present value of the losses given default. To keep the analysis still relatively simple, we assume that the number of times when the default events may happen is not infinite, but finite; to this end we divide the period $[t,T]$ in N smaller periods, at the end of each either (or both) defaults may occur. Let T_n, for $n = 1, ..., N$ be the times (with $T_N = T$): we generalise equation as follows:

$$\mathbf{EL}(t,T) = \mathbf{EAD} - \sum_{n=1}^{N} \left[v_{\mathbf{Coll}}(T_n) + [P_1 \mathbf{PD}_d^I(T_{n-1},T_n) + P_2((1-p_d)\mathbf{PD}_d^I(T_{n-1},T_n) + p_d)]\mathbf{Lgd} \right] / \mathbf{PD}_d(T_{n-1},T_n) \tag{12}$$

where

$$P_1(T_{n-1},T_n) = (1 - \mathbf{PD}^C(T_{n-1},T_n))\mathbf{PD}^I_{\mathbf{Coll}}(T_{n-1},T_n)$$

and

$$P_2(T_{n-1},T_n) = \mathbf{PD}^C(T_{n-1},T_n)((1-p_{\mathbf{Coll}})\mathbf{PD}^I_{\mathbf{Coll}}(T_{n-1},T_n) + p_{\mathbf{Coll}})$$

The haircut is defined similarly to what shown in equation (10). The resulting haircut is an average of the expected losses suffered in the event of default occurring on one of the possible dates T_n.

The **EL** defined as in (12) is strongly resembling the **CVA** at the inception t of the repo transaction, evaluated by the buyer (lender):

$$\mathbf{CVA}^l(t,T) = \mathbf{E}^Q\left[\int_t^T \mathcal{D}(t,u)\max\left[(1-\mathcal{H}(t))V(t)\times[1+r^E\times\tau^E(t,T)]-V_{\mathbf{Coll}}(u),0\right]d\mathbf{PD}_d(t,u)\right] \tag{13}$$

where $\mathcal{D}(t,u) = \exp(-\int_t^u r_v dv)$ is the risk-free discount factor at time t for cash-flows occurring at time T_n. All the probabilities of default are computed within the intervals included in the period $[t,T]$.

Assuming we use the same risk-neutral measure to compute the **EL** and to compute the **CVA**,[8] the main difference between the two measures is that **EL** is the value of the expected losses given the default of the borrower, whereas the **CVA** is the present value of the expected (unconditional) losses due to the counterparty credit risk. Besides it is also worth noting that **EL** is computed at a minimum level (at the chosen confidence level) of the collateral bond's value over the life of the contract, whereas the $\mathbf{CVA}^l$ considers only the expected values. As such $\mathbf{EL} \geq \mathbf{CVA}^l$ and $\mathbf{EL} = 0$ (after setting the fair haircut as shown above) implies *a fortiori* that the $\mathbf{CVA}^l = 0$ as well.

2.3 Margin Maintenance

Both counterparties involved in a repo transaction (*i.e.*: the buyer and seller), are exposed to the risk that the market value of the collateral asset may fall below or rise above the repurchase price, thus originating a counterparty credit exposure for, respectively, the buyer or the seller. This counterparty credit exposures is eliminated by a transfer of margin to the exposed party by the other party: collateral can be posted either by a cash payment or by a transfer of additional collateral assets.

In practice a margin call is made when one party has a net exposure to the other. The net exposure is given by the sum of all transaction exposures referring to each repo contract existing between the two parties (see section 4(c) of GMRA 2000 [3] and 2011 [2]), plus any income due from the other party but unpaid (*i.e.*: manufactured payments and interest payments) plus net margin still held by the other party. The transaction exposure can be defined in two ways, depending on whether an initial margin or a haircut has been defined in the contract (see section 2(ww) of GMRA 2000 [3] and section 2(xx) 2011 [2]) and it is basically a sort of mark to the market of the value of the contract on a given date before the expiry. As a best practice, net exposure should be calculated at least every business day.

[8]One may actually use a real world measure to calculate **EL** and hence set the fair haircuts. In this case additional differences may arise between the two measures.

Transaction exposure when the initial margin is used.

Let us start with the case the repo contract provides for an initial margin: the transaction exposure (of the repo buyer) $\mathbf{TE}(s)$ on a date s, with $s < T$ is defined as:

$$\mathbf{TE}(s) = \Big(\mathbf{RP}(s)\mathbf{IM}\Big) - \mathbf{MV}(s) \tag{14}$$

where $\mathbf{IM}$ has been defined before, and:

- $\mathbf{RP}(s)$ is the repurchase price at time s, and it has computed as:

  $$\mathbf{RP}(s) = \mathbf{PP}\Big(1 + r^E \times \tau^E(t,s)\Big)$$

  with r^E the contract repo rate, and $\tau^E(t,s)$ is the year fraction of the period $[t,s]$ calculated according to the day count and basis convention of the contract;

- $\mathbf{MV}(s)$ is the market value of the collateral, which in case of a bond is:

  $$\mathbf{MV}(s) = N_{\mathbf{Coll}} \times \left(\frac{P(s) + c \times \tau^B(t_{-1},s)}{100}\right)$$

  with $P(s)$ the bond market (clean) price, c the coupon rate and $\tau^B(t_{-1},s)$ the year fraction (computed according to the proper conventions) of the period $[t_{-1},s]$, where t_{-1} is the last coupon payment date.

It is worth stressing that the repurchase price $\mathbf{RP}(s)$ is not the contract price, but the theoretical price that the seller should pay to buy back the bond, should the contract be terminated at time s: the accrued interests run from the start of the contract t to s.[9]

Transaction exposure when the haircut is used.

If in the repo contract a haircut is applied on the purchase price of the collateral asset, then the transaction exposure (of the repo buyer) $\mathbf{TE}(s)$ on a date s, with $s < T$ is:

$$\mathbf{TE}(s) = \mathbf{RP}(s) - \Big(\mathbf{MV}(s)[1 - \mathcal{H}(t)]\Big) \tag{15}$$

where $\mathcal{H}$ is the haircut and has been defined before, and the repurchase price at time s, $\mathbf{RP}(s)$, and the market value of the collateral, $\mathbf{MV}(s)$, are defined as in the case before when the initial margin is used.

Remark 2.1. *It should be noted that the variations of the collateral depend only on the changes of its market value. In reality the collateral should cover the expected losses given the default of the borrower. When the borrower's and issuer's defaults are uncorrelated, provided that the market price of the collateral includes also the variations of the issuer's* **PD**, *then the margin maintenance works quite well. When the wrong-way risk is relevant (high correlations between the two defaults), then the mechanism, which does not consider any variation of the borrower's* **PD**, *is not able to properly adjust the level of collateral.*

[9] In reality, in counting the number of days, one should stop at the day before the margin call is made. So the number of days is up to, but excluding, the margin delivery date.

The collateral $C^l(s)$ that the lender (buyer) should receive at time s is equal to $\mathbf{TE}(s)$. Considering the collateral already received by the buyer, the variation between two margin calls in times q and s, $(q < s)$ is:

$$\Delta C^l(s) = \mathbf{TE}(s) - C^l(q) \tag{16}$$

Negative variations and values means the buyer posts instead of receiving collateral.

If the repo contract is assisted by a maintainance margin agreement, the $\mathbf{CVA}^l$ the lender should compute is:

$$\mathbf{CVA}^l(t,T) = \mathbf{E}^Q\left[\int_t^T \mathcal{D}(t,u) \max\left[(1-\mathcal{H}(t))V(t) \times [1 + r^E \times \tau^E(t,T)] - V_{\mathbf{Coll}}(u) - C^l(u), 0\right] d\mathbf{PD}_d(t,u)\right] \tag{17}$$

Example 2.3. *The calculation of the transaction exposure for the repo contract in Example 2.1 is shown in this example, starting with the recapitulation the details of the contract in Table 8.*

	Haircut	Initial Margin
Amount	1,000,000.00	1,000,000.00
Dirty Price Price Collateral	115.32	115.32
N. Days	92	92
Repo rate	1.75%	1.75%
Initial Margin	-	3%
Haircut	3%	-
Market Value of Collateral	1,153,212.33	1,153,212.33
Cash Received	1,118,615.96	1,119,623.62
Cash Returned	1,123,618.66	1,124,630.83

Table 8: Summary of the repo contract details presented in Example 2.1.

At the inception of the contract $t = 0$, or Day 0, as indicated in Table 9, the exposure is obviously nil, as resulting by applying formulae (14) *and* (15) *by inputting the starting dirty price of the collateral bond. At day 1 we suppose that the dirty price collapses at 114.00, which should imply an increase of the transaction exposure to the buyer: actually this is what happens as shown in the table. If the haircut and the initial margin have the same value (3% in this example), the variation of the transaction exposure is smaller where the haircut is used in stead of the initial margin.*

Types of Collateral That Can Be Posted after a Margin Call

The margin call can fulfilled by posting collateral to the counterparty either in cash or in other securities. The following rules are followed:

- where the margin is posted in cash, the party receiving collateral should remunerate it at a suitable reference rate. As an example, if the transaction exposure is checked daily and the margin is called with this frequency, then an O/N rate (such as the EONIA for collateral denominated in Euro) should be applied;

Exposure Day 0	Haircut	Initial Margin
Dirty Price Collateral	115.32	115.32
Market Value of Collateral	1,153,212.33	1,153,212.33
Repo Price Time 0	1,118,615.96	1,119,623.62
Transaction Exposure	-	-
Exposure Day 1		
Dirty Price Collateral	114.00	114.00
Market Value of Collateral	1,140,000.00	1,140,000.00
Repo Price Time 1d	1,118,670.34	1,119,678.05
Transaction Exposure	12,870.34	13,268.39

Table 9: Variation of the transaction exposure from the trading date (Day 0) to the following date (Day 1), in case where the haircut or the initial margin is used.

- where the margin is posted in securities, the value considered is their dirty price (clean " market" price plus accrued interests); if interests are paid by the securities during the life of the contract, they should be immediately transferred to the party posting the collateral. Moreover, if an initial margin is set on the collateral security delivered at the inception of the repo, then a margin is applied also on the securities posted as collateral on margin calls: the haircut can be different from the one set at inception even if the security posted on margin call is the same as the repo collateral, since market conditions might have changed after the purchase date.

2.4 Valuation

We show here how to evaluate a repo contract, following the incremental valuation approach outlined in Castagna [6] assuming that the agent is a market-maker (or a hedger): basically the approach implies that when a contract is evaluated considering its inclusion in the existing market-maker's balance sheet, if the default of the counterparty is not triggering also its own default, then the market-maker should evaluate the contract by considering just the counterparty credit risk (**CVA**) and the funding costs paid to hedge (replicate) the contract (**FVA** and **LVA**)[10]. The debit valuation adjustment never plays a role, whereas the feasibility to attain a funding benefit due to possible positive cash-flows should be verified on a case-by-case base: excluding the funding benefit is the simplest and safest policy.

[10]See Castagna [4] for a definition of **LVA**.

Valuation for the Lender (Buyer)

Let us start in a simplified setting and assume that the market-maker is a repo buyer (lender) which is not subject to credit risk (*i.e.*: its funding spread is equal to zero); to keep the analysis simple we also assume that the contract does not provide for any maintenance margin. In very general terms, the valuation formula for 1 unit notional can be written as:

$$\mathbf{Repo}^l(t,T) = \mathbf{E}^Q\Big[-\mathbf{PP} + \mathcal{D}(t,T)\mathbf{RP}\Big] - \mathbf{CVA}^l(t,T) + \mathbf{LVA}^l(t,T) \tag{18}$$

The $\mathbf{LVA}^l(t,T)$ is formally defined as:

$$\mathbf{LVA}^l(t,T) = \mathbf{E}^Q\left[\int_t^T \mathcal{D}(t,u)C^l_{\mathbf{Cash}}(u)(r_u - c_u)du\right] \tag{19}$$

where c is the rate at which the collateral posted in cash ($C^l_{\mathbf{Cash}}$) is remunerated. If we assume that the risk-free rate is also the remuneration rate, ad it happens in practice when c is indexed at the O/N rate, which is conventionally assumed to be also the risk-free rate, than the $\mathbf{LVA}^l(t,T) = 0$.

We discussed above that if the haircut set at the inception t is the fair one, than the $\mathbf{CVA}^l(t,T)$ is zero: we can write equation (20)

$$\mathbf{Repo}^l(t,T) = \mathbf{E}^Q\Big[-\mathbf{MV}(1-\mathcal{H}(t)) + \mathcal{D}(t,T)\big[\mathbf{MV}(1-\mathcal{H}(t)) \times (1 + r^E \times \tau^E(t,T))\big]\Big] \tag{20}$$

The fair repo rate is the level of r^E that makes zero the value of $\mathbf{Repo}(t,T)$:

$$r^E = \left(\frac{1}{P^D(t,T)} - 1\right)\frac{1}{\tau^E(t,T)} \tag{21}$$

where $P^D(t,T_n) = \mathbf{E}^Q\Big[\exp(-\int_t^T r_s ds)\Big]$ is the price of a risk-free zero-coupon bond. Equation (21) is equal to the simply-compounded risk-free for the period equal to the duration of the repo contract. So the repo, when the haircut is fairly set, should not be evaluated differently from a deposit the lender closes with the borrower, as if the latter were a risk-free counterparty (even though actually it is not).

This result is perfectly consistent with a no-arbitrage argument in evaluating the repo, when the evaluator does not consider the counterparty's default and does not pay a funding spread in the market. The argument is the following: assume the repo rate is above the risk-free rate for a given maturity: the buyer (lender) can take the cash in the money market and use that to pay the purchase price of the repo: the repurchase price cashed in at the expiry is enough to pay back the debt plus interests and leave also an arbitrage profit. To avoid the arbitrage, the repo rate should not be above the risk-free rate.

In reality the lender is not a risk-free agent, so it is likely that it pays a spread when funding the cash needed when closing the contracts, in case it does not have a positive cash amount available. The inclusion of the funding costs within the evaluation of the repo entails adding the **FVA** component to equation (20), which is a cost and as such it abates the value of the contract to the buyer, so it enters with the negative sign:

$$\mathbf{Repo}^l(t,T) = \mathbf{E}^Q\Big[-\mathbf{PP} + \mathcal{D}(t,T)\mathbf{RP}\Big] - \mathbf{CVA}^l(t,T) - \mathbf{FVA}^l(t,T) + \mathbf{LVA}^l(t,T) \quad (22)$$

The **FVA** is defined as:[11]

$$\mathbf{FVA} = P^D(t,T)\mathbf{MV}(1-\mathcal{H}(t)) \times s^l \times \tau^E(t,T) \quad (23)$$

where s^l is the (constant) funding spread paid over the risk-free rate by the lender, over the period corresponding to the duration of the contract. The funding cost to the lender is originated by the funding spread paid on the cash it has to raise to lend money to the borrower.

If the haircut is fair (i.e.: $\mathbf{CVA}^l = 0$), the repo rate including the lender's funding is:

$$\begin{aligned} r^E &= \left[\left(\frac{1}{P^D(t,T)} - 1\right) + \frac{1}{P^D(t,T)}\frac{\mathbf{FVA}^l(t,T)}{\mathbf{MV}(1-\mathcal{H}(t))}\right]\frac{1}{\tau^E(t,T)} \\ &= r + s^l \end{aligned} \quad (24)$$

where we has assumed that the funding costs originated by the margin calls is enough small to be considered negligible.

In this case the no-arbitrage argument should be refined by setting the upper limit to the repo rate at the funding rate (*i.e.*: risk-free rate plus funding spread) of the lender, or more generally to the average funding rate paid in the money market by the market-makers/lenders that can enter in reverse repo transactions. This is the conclusion also in Duffie [8].

Valuation for the Borrower (Seller)

From the bowrrower (seller) point of view, the value of the repo is:

$$\mathbf{Repo}^d(t,T) = \mathbf{E}^Q\Big[+\mathbf{PP} - \mathcal{D}(t,T)\mathbf{RP}\Big] - \mathbf{CVA}^d(t,T) \quad (25)$$

where the **CVA**, from the seller (borrower) perspective, is defined as:

$$\begin{aligned} \mathbf{CVA}^d(t,T) = \mathbf{E}^Q\Bigg[\int_t^T \mathcal{D}(t,s)\max\big[&V_{\mathbf{Coll}} \\ &- (1-\mathcal{H})V(t) \times [1 + r^E \times \tau^E(t,T)] - C^d(s), 0\big]d\mathbf{PD}_l(t,s)\Bigg] \end{aligned} \quad (26)$$

and $\mathbf{PD}_l(t,s)$ is the probability of default of the lender between t and s; $C^d(s)$ is the collateral received by the borrower from the maintainance margin mechanism described

[11] The definition of the **FVA** could be made more precise by including the possibility to pay the funding spread for a shorter period than $\tau^E(t,T)$ by including in the formula the default of the borrower. We do not consider here such case.

above ($C^d(s) = -C^l(s)$). It is quite easy to check that if $\mathbf{CVA}^l = 0$ (*i.e.*: the haircut is fairly set), $\mathbf{CVA}^d \geq 0$, the equality holding only if the lender's default probability is zero. It is finally worth stressing that the $\mathbf{FVA}^d$ is implicitly determined by the lender's funding spread. More specifically, assuming for a moment that the lender is default-risk free (*i.e.*: $\mathbf{CVA}^d = 0$), the repo is worth:

$$\begin{aligned}\mathbf{Repo}^d(t,T) &= \mathbf{E}^Q\left[+\mathbf{PP} - \mathcal{D}(t,T)\mathbf{RP}\right] = \mathbf{PP} - P^D(t,T)(1-\mathcal{H})V(t) \times [1 + r^E \times \tau^E(t,T)] \\ &= \mathbf{PP} - (1-\mathcal{H})V(t) - P^D(t,T)(1-\mathcal{H})V(t) \times s^l \times \tau^E(t,T) \\ &= -P^D(t,T)(1-\mathcal{H})V(t) \times s^l \times \tau^E(t,T) = -\mathbf{FVA}^d(t,T)\end{aligned} \tag{27}$$

Equation (27) states that the value of the repo to the cash borrower is negative and equal to the lender's funding costs, or $\mathbf{FVA}^d(t,T) = \mathbf{FVA}^l(t,T)$: if the agent lending cash is on the strong side in the bargaining power, it is able to charge its funding costs to the borrower, which will eventually pay the lender's funding spread s^l instead of its own (unsecured) funding spread s^d.[12] The repo allows then the borrower to lower its funding costs buy aligning them to those of the credit-worthier lender. Hence the implied funding cost to borrow cash via a repo transaction (secured funding) is just the repo rate $r^E = r + s^l$.

If the lender is not a default-risk free agent, then the repo's value to the borrower should include also the counterparty credit risk costs:

$$\mathbf{Repo}^d(t,T) = -\mathbf{FVA}^d(t,T) - \mathbf{CVA}^d(t,T) \tag{28}$$

so that the value of the repo is even more negative than the case examined above. In this case, the implied funding rate paid by the borrower is:

$$r + s^E = r + s^l + \frac{1}{P^D(t,T)} \frac{\mathbf{CVA}^l(t,T)}{\mathbf{MV}(1-\mathcal{H}(t))} \frac{1}{\tau^E(t,T)} > r^E \tag{29}$$

Hence the funding spread is higher than the lender's funding spread, when the borrower has to accept to pay the $\mathbf{CVA}^l$ if it wants to receive the money. In any case, it will also be very likely that, when the lender has a high credit standing, the implied funding is lower than the borrower's unsecured funding cost.

3 The Sell/Buy Back

Definition 3.1. *A* ***sell/buy back*** *transaction is virtually identical to a repo transaction: the legal title of the collateral is transferred by the seller to the buyer, against a payment of cash; the seller agrees to buy back the same (or equivalent) collateral at a future date at a specific price.*

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[12] Under a strict theoretical point of view, since we assumed the repo lender (buyer) is risk-free, its funding spreas s^l should be zero.

The main differences between a repo and a sell/buy back are that the latter may also not be documented under a master agreement, but it can be simply dealt as two separated trades. A direct consequence of this possibility is that, for undocumented sell/buy-backs, it is not possible to set up a (both maintenance and initial) margin process, a haircut at the start, or close-out and set-off if default by either party occurs. Finally, no manufactured payments occur during the life of a sell/buy back: all payments made by the collateral asset are included into the repurchase price paid on the repurchase date.

Nowadays, sell/buy backs are simply seen as a form of repo, as such they are documented in the Buy/Sell Back Annex to the new 2011 GMRA agreement (see [2]). Also the quotation is now similar to repo contracts, *i.e.*: in terms of contract rate[13] instead of forward bond price, as it used to be in the past. The difference between a repurchase agreements and documented sell/buy-backs is basically only due the method the latter use to mitigate credit exposures originated by the volatility of the collateral asset's market price. In more details, repurchase agreements use the margin maintenance mechanism we have described above (see Section 2.3) to restore the equivalence between the the values of cash lent and collateral received; documented sell/buy-backs mitigate credit exposures by terminating the transaction and simultaneously opening a new transaction for the residual time to maturity, keeping the same terms as the original transaction, but re-aligning the value of the cash to the new market value of the collateral asset. We will briefly describe this process, denominated "re-repricing", below.

3.1 Purchase and Repurchase Price

Similarly to a repo transaction, the seller of the collateral asset at inception receives the purchase price; the final cash paid at the expiry by the seller, to buy the collateral asset back, considers also the interim payments made by the asset compounded at the contract repo rate. Since the mechanics is quite similar to the repo case, the purchase and repurchase price are determined exactly as in equation (1) ((2) if an initial margin is applied instead of a haircut).

If the sell/buy back is quoted in terms of the repo rate, we have everything it is needed. If, on the contrary, the contract is quoted in terms of collateral asset price, then we need to determine the initial and forward price at which, respectively, it is sold and bought back. The initial price is simply the price of the asset prevailing in the market when the contract starts in t. When the collateral asset is a bond, the clean price is considered, even though the cash exchanged is the market value of the position, which included the accrued interests.

The forward price is set in terms of clean price too, so that accrued interests must be deducted. Moreover, all the interim payments have to be compounded at the contract repo rate and deducted, since they have to be returned at the original seller. In formula, we have that the forward clean price $P(T)$, where the haircut is applied, is equal to:

$$P(T) = \frac{\mathbf{RP}(T)}{N_{\mathbf{Coll}}} \times \frac{1}{1 - \mathcal{H}(t)} - \mathbf{AI}(t, T) - \mathbf{IP} \tag{30}$$

[13] The contract rate is named repo rate also for sell/buy back transactions.

where $\mathbf{AI}(t,T) = c \times \tau^B(t_{-1},T)$ is the bond's accrued interest from the last coupon date to the expiry T, and **IP** is the sum of the compounded interim payments $C(t_i)$:

$$\mathbf{IP} = \sum C(t_i) \times [1 + r^E \tau(t_i, T)] \tag{31}$$

Where the initial margin is applied, the forward clean price is:

$$P(T) = \frac{\mathbf{RP}(T)}{N_{\mathbf{Coll}}} \times \mathbf{IM} - \mathbf{AI}(t,T) - \mathbf{IP} \tag{32}$$

Example 3.1. *Assume the collateral asset is the bond in example 2.1 and that the sell/buy back has the same expiry and rate as the repo therein analysed. If the sell/buy back is quoted in terms of repo rate, then the same calculations for the initial purchase price and the final repurchase prices apply.*

If the sell/buy back is quoted in terms of bond's price, then the terms of the contract will refer to the initial and forward price. The initial price is the clean price as in the example 2.1, i.e.: 115.05; the dirty price is 115.3212, since accrued interests are 0.2712. The initial cash exchanged is the same as for a standard repo, depending on whether a haircut or an initial margin is applied: in table 10 we show the initial cash in both cases.

	Haircut	Initial Margin
Purchase Price	111.8616	111.9624
Sell/Buy Back Cash	1,118,615.96	1,119,623.62
Bond Notional	1,000,000.00	1,000,000.00
Repo Rate	1.75%	1.75%
Maturity	06/11/2014	06/11/2014
Days	92	92
Repo Interests	5,002.70	5,007.21
Cash Returned	1,123,618.66	1,124,630.83
Accrued (92 days)	1.03	1.03
Repurchase Price	114.8096	114.8096

Table 10: Details of a sell/buy back when a haircut or an initial margin is applied.

The forward price for the haircut is determined by means of formula (30)*:*

$$P(T) = \frac{1,123,618.66}{1,000,000.00} \times 100 \times \frac{1}{1 - 3\%} - 1.03 = 114.8096$$

If the initial margin is applied, using formula (32) *will yield the same result.*

3.2 Transaction Exposure and Re-pricing

The transaction exposures on sell/buy-backs are not cancelled by terminating the original transaction and simultaneously starting a new transaction with the same maturity, in which two possible alternatives can be adopted (see 4(j) and 4(k) of the GMRA 2000 [3] and 4(k) and 4(l) of the GMRA 2011 [2]):

- the purchase price of the new transaction is set equal to the new Market Value of the collateral asset or

- the nominal value of the asset is changed to re-align the new market value to the original purchase price.

In the first case the net difference between the repurchase price of the terminated transaction and the purchase price of the new one is paid; in the second case the net difference between the original amount of collateral asset and the new amount is delivered.

In the GMRA the first method is called "re-pricing" and the second method is called "adjustment". Under the re-pricing method, accrued repo interest is paid to the buyer and it is not included in the new purchase price.

Finally, when margining is applied to a portfolio of transactions, the re-pricing, or the adjustment, is applied to the single transactions. It is customary to reprice or adjust transactions in sequence, starting with the transaction with the highest transaction exposure.

Assume we are in that the sell/buy back started in t and expires in T; if the "re-pricing" method is used, at time $t < s < T$, the new deal replacing the latest one has a new purchase price $\mathbf{PP}^*$ (replacing the old $\mathbf{PP}$) such that:

$$\mathbf{PP}^* = \frac{\mathbf{MV}(s)}{\mathbf{IM}} \tag{33}$$

or

$$\mathbf{PP}^* = \mathbf{MV}(s) \times [1 - \mathcal{H}(t)] \tag{34}$$

depending on whether an initial margin or a haircut was set in the original contract (the notation is the same as the one used before for repos). The repurchase price of the new transaction is s is equal to the market value of the collateral at time s.

If the "adjustment" method is used, then at time s the new deal will have an adjusted notional such that:

$$\mathbf{MV}^*(s) = \mathbf{RP}(s) \times \mathbf{IM} \tag{35}$$

or

$$\mathbf{MV}^*(s) = \frac{\mathbf{RP}(s)}{1 - \mathcal{H}(t)} \tag{36}$$

again, depending on whether an initial margin or a haircut was set in the original contract. In this case, the equivalence in (35), or (36), is restored by changing the notional of the deal from $N_{\mathbf{Coll}}$ to $N^*_{\mathbf{Coll}}$, so that the market value of the collateral is:

$$\mathbf{MV}^*(s) = N^*_{\mathbf{Coll}} \times \left(\frac{P(s) + c \times \tau^B(t_{-1}, s)}{100} \right)$$

Example 3.2. *We refer to the bond data in examples 3.1 and 2.3 and we show how the credit risk mitigation mechanism of a sell/buy back works in practice. To this end, consider the variation in the dirty price of the collateral bond after 1 day from 115.32 to 114.00. The sell/buy back is "adjusted" by determining the new notional amount to collateral to deliver in the new transaction cancelling the old one. In table 11 we show the result in case a haircut or an initial margin is applied. The notional amount of collateral of the*

	Haircut	Initial Margin
Adjusted Collateral Notional	1,011,638.94	1,011,638.94
Cash Returned	1,123,618.90	1,124,631.07
Accrued	1.03	1.03
New Clean Price	113.48	113.48

Table 11: "Adjustment" of a sell/buy back contract.

new transaction is clearly the same in both cases and it is the result of the application of formulae (35) *and* (36)*.*

The forward clean price, in case the sell/buy back is quoted in terms of price, is also the same in both cases and it is determined as:

$$P(T) = \frac{1,123,618.90}{1,000,000.00} \times 100 \times \frac{1}{1-3\%} - 1.03 = 113.48$$

if a haircut is applied. When an initial margin is applied, the forward price is

$$P(T) = \frac{1,123,618.90}{1,000,000.00} \times 100 \times (1+3\%) - 1.03 = 113.48$$

The transaction exposures, from table 9, are 12,870.34 and 13,258.39 respectively for the case a haircut or an initial margin is applied. It should be noted that the transaction exposure are the same for a repo and a sell/buy back, if the two contracts have the same economic terms. The "adjustment" mechanism is able to reset at zero the exposure. Consider the case of a haircut: the difference in the notional between the old and new contract is: $1,011,638.94 - 1,000,000.00 = 11,638.94$ *(see table 11 first column). In practice there is a netting of cash-flows between the seller and the buyer of the collateral asset, so that an amount of:*

$$11,638.94 \times 114.00/100 \times (1-3\%) = 12,870.34$$

is paid by the seller to the buyer, exactly compensating the latter (i.e.: the lender) for the increase of the transaction exposure. A similar check can be made for the initial margin case: the cash exchanged is:

$$11,638.94 \times 114.00/100 = 13,258.39$$

The alternative mechanism is the "re-pricing": in this case the notional is kept constant while the price is changed as shown in table

	Haircut	Initial Margin
Repriced PP	1,105,800.00	1,106,796.12
Cash Returned	1,110,624.62	1,111,625.08
Accrued	1.03	1.03
New Clean Price	113.47	113.47

Table 12: "Repricing" of a sell/buy back contract.

The new purchase price is:

$$1,400,000.00 \times (1 - 3\%) = 1,105,800.00$$

when a haircut is applied. The cash settled between the buyer and the seller is in this case simply the difference between the updated purchase price of the old deal (1,118,615.96 see table9) and the purchase price of the new deal (1,105,800.00), so:

$$1,118,615.96 - 1,105,800.00 = 12,870.34$$

so that the transaction exposure is fully compensated.

When the initial margin is applied, the new purchase price is:

$$\frac{1,400,000.00}{1 + 3\%} = 1,106,796.12$$

and the compensated transaction exposure is:

$$(1,118,615.96 - 1,105,800.00) \times (1 + 3\%) = 13,258.39$$

3.3 Fair Haircut and Valuation

The fair haircut of a sell/buy back can be calculated by the same approach as the one sketched above for a repo. Also the valuation of a sell/buy back is virtually identical to the valuation of a repo contract analysed before. Once the haircut is fairly set and a credit risk mitigation mechanism is chosen, the **CVA** can be set equal to zero also in this case.

4 Securities Lending

Definition 4.1. *At the inception of a securities lending transaction, the Lender transfers title to a single security, or a basket of securities to the* ***borrower****, receiving in exchange either the title to another security (or a basket of securities) or alternatively cash, and the payment of a fee; at the expiry of the contract, or on demand, the* ***lender*** *will transfer title to equivalent collateral or repay the cash plus an agreed return, in exchange for title to the security, or a basket of securities, equivalent to the one it transferred at the inception.*

The security (or basket of security) or cash received by the lender at the start of the contract is the collateral the borrower posts to receive title to the lent security (or basket of securities). The denomination of the contract and the wording used in the definition may be misleading, since the actual title to security is transferred from the lender to the borrower, as in a repo transaction.[14] Figure 2 provides a visual summary of the obbligations performed by both parties in the contract.

The effects of a securities lending transaction are basically analogous to those of a repo contract, the main differences lying in the exchange between the two parties, which

[14] Although this statement is generally true, it worth noting that the transfer of the title may not occur in the US market.

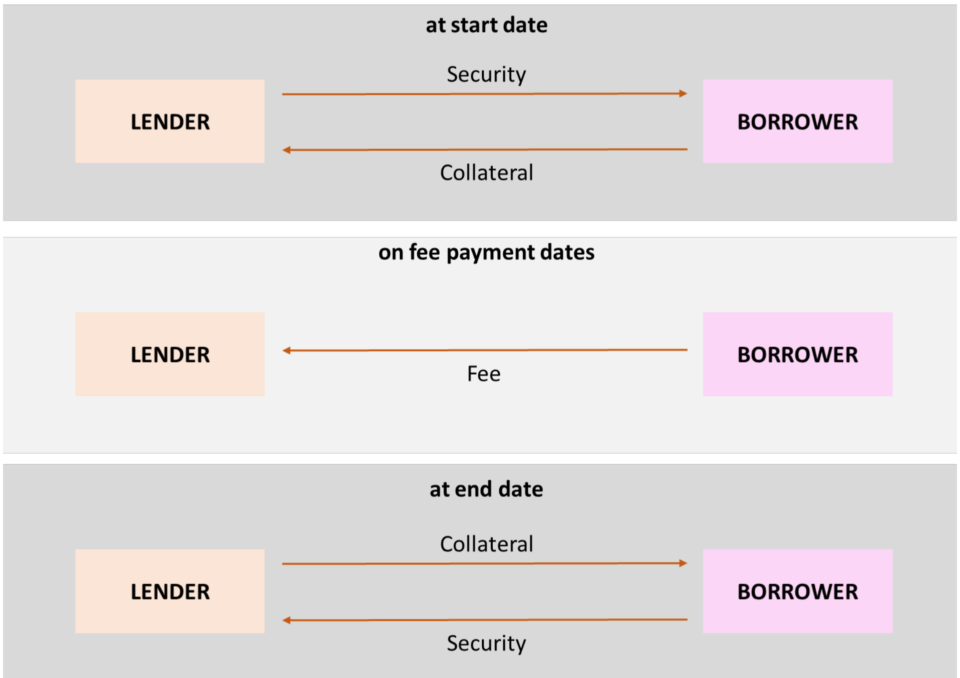

Figure 2: Obligations of a security lending contracts.

can be a security against another security in the securities lending, whereas it is against cash in a repo. Another relevant difference is that the expiry of a repo contract is defined, while a securities lending is generally dealt on an open basis. Open basis transactions may be terminated by the borrower, when it returns securities, or by the lender, when it recalls them. Similarly to a repo, all payments made by the lent security are transferred to the lender by means of a "manufactured" payment by the borrower. Analogously, when non-cash collateral is delivered, any payment made by the collateral security is transferred to the borrower by the lender.

Under an economic point of view, a securities lending transaction is typically security-driven, which means that the demand of the borrower for a specific security (or a basket of securities) is the main reason for the deal. This is also reflected by the terms denoting the parties, which are chosen with reference to the security rather than the cash: the borrower receives cash against securities in a repo, it receives securities against cash (or other securities) in a securities lending. Furthermore, the securities lender receives a fee contrarily to what happens in a repo, where the seller (cash borrower) pays an interest rate: this happens when the lender take securities as collateral; when it is given cash as collateral, it pay the borrower an interest but at a rate (the rebate rate) that is lower than market rates. So, in any case, the economic rationale produces a yield for the securities lender.

For securities lending, the standard master agreement adopted by institutional market agents is the ISLA GMSLA [1].

4.1 Quotation: Fee and Collateral Margin

The quotation of a securities lending contract requires the definition of a fee to be paid to the (securities) lender on a periodic basis. Typically, the fee will be paid monthly at the end of each month of the life of the contract; also the final fee referring to the month in which the contract either expires or it is recalled (in open ended transactions) is paid at the last day of the month.

A margin is applied to the value of the collateral delivered by the borrower against the lent securities: the margin is defined as in the repo contract case analysed in section 2.2. It has the same purpose as in the repo case to protect the (securities) lender from the default of the (securities) borrower.

4.2 Fair Margin

The security lender wishes to set at inception a margin **IM** such that the expected loss suffered in the event of the borrower's default is nil. To set the fair level of initial margin, we can adopt an approach similar to the one sketched above for a repo contract (also for the notation, we refer to the one used above). Assume at the start of the contract $t = 0$, a time horizon T: this time may clash with the expiry of the contract, or an expected duration if it has been dealt on an open basis.[15] Let $S(T)$ be the value of the lent security in T, and let $v_{\mathbf{Coll}}(T)$ be the minimum market value of the posted collateral in T, calculated at a given confidence level, say 99%. The fair margin is set so that the expected loss **EL** on the borrower's default is:

$$\mathbf{EL}(t,T) = \mathbf{E}\left[\max[S(T) - v_{\mathbf{Coll}}(T)\mathbf{IM}, 0]\middle| D_d(t,T) = 1\right] \tag{37}$$

The notation is the same as in section 2.2 ($D_d(t,T)$ is the stochastic variable equal to 1 if the security borrower goes bankrupt in the time interval $[t,T]$). The main difference between equations (37) and (7) is that in the former the $\mathbf{EAD} = S(T)$, and is a stochastic variable, whereas in the latter **EAD** equals the amount of money lent in the repo, so that it is a given quantity. We will tackle the problem with the stochastic **EAD** similarly to the uncertainty of the value of the collateral: we will calculate the maximum exposure at a given confidence level, say 99%. Let $S^{\alpha}(T)$ be this level, when α is the chosen confidence level.

Assuming independence between the event of default of the issuer of the asset underlying the security lending contract, and the default of the collateral asset, (and, more generally, any correlation between the values of the two assets), following the reasoning in section 2.2, we can write (37) that:

$$\mathbf{EL}(t,T) = \mathbf{E}\left[S^{\alpha}(T) - v_{\mathbf{Coll}}(T)\mathbf{IM}\middle| D_d(t,T) = 1\right] \tag{38}$$

The equation (38) states that the expected loss is equal to the expected positive difference between the maximum value of the lent asset and the minimum value of the collateral,

[15] We recall here that both parties have the right to terminate the contract by giving a notice to the other party, when it was dealt on an open basis.

times the initial margin, in T, given the collateral issuer's default indicator function ($D_{\mathbf{Coll}}(t,T)$) is 1, or the default occurs.

Given the independence assumption, the expected (maximum) exposure is $\mathbf{E}[S^{\alpha}(T)|D_d(t,T)=1]=S^{\alpha}(T)$, while the expected (minimum) value of the collateral is:

$$\mathbf{E}[v_{\mathbf{Coll}}(T)\mathbf{IM}|D_d(t,T)=1]=$$
$$\mathbf{E}[v_{\mathbf{Coll}}(T))\mathbf{IM}(1-D_{\mathbf{Coll}}(t,T))+(v_{\mathbf{Coll}}(T)-\mathbf{Lgd})\mathbf{IM})D_{\mathbf{Coll}}(t,T)\Big|D_d(t,T)=1]$$

or, more explicitly:

$$\begin{aligned}\mathbf{EL}(t,T)=&S^{\alpha}(T)-v_{\mathbf{Coll}}(T)\mathbf{IM}\\&+\left[[P_1\mathbf{PD}_d^I(t,T)+P_2((1-p_d)\mathbf{PD}_d^I(t,T)+p_d)]\mathbf{LgdIM}\right]/\mathbf{PD}_d(t;T)\end{aligned} \quad (39)$$

The initial margin $\mathbf{IM}$ is the level that makes $\mathbf{EL}(t,T)=0$, or:

$$\mathbf{IM}=\frac{S^{\alpha}(T)}{v_{\mathbf{Coll}}(T)-\left[[P_1\mathbf{PD}_d^I(t,T)+P_2((1-p_d)\mathbf{PD}_d^I(t,T)+p_d)]\mathbf{Lgd}\right]/\mathbf{PD}_d(t;T)} \quad (40)$$

4.3 Collateral Maintenance

Collateral has to be marked to the market during the life of the contract: on any business day, the aggregated market value of the collateral delivered to, or deposited with, the security lender (**Posted Collateral**) must be equivalent to aggregated market value of the securities, considering also the applicable margin (**Required Collateral Value**). In case netting is excluded, the provision applies on a single contract basis, instead of an aggregated basis. Limiting the analysis to the latter case, for a given contract we have in formula that:

$$V_{\mathbf{Coll}}(s)\mathbf{IM}=S(s)\mathbf{IM} \quad (41)$$

where, at time s, $V_{\mathbf{Coll}}(s)\mathbf{IM}$ is the value of the posted collateral and S is the value of the lent security ($\mathbf{IM}$ is the margin applied). If on any business day $V_{\mathbf{Coll}}(s)\mathbf{IM}<S(s)\mathbf{IM}$, additional collateral is posted by the security borrower so as to reestablish the equivalence. On the contrary, if $V_{\mathbf{Coll}}(s)>S(s)\mathbf{IM}$, excess collateral is returned by the lender to the borrower.

The collateral maintenance mechanism allows to set the $\mathbf{IM}$ by considering just a margin period of risk, that is the period between the default and its actual recognition. Typically this period is set equal to 10 business days. Once the initial margin is computed, the collateral maintenance mechanism warrants a nil expected loss ($\mathbf{EL}(t,T)=0$), for any time horizon.

The collateral $C^l(s)$ that the security lender should receive at time s is equal to $S(s)\mathbf{IM}-V_{\mathbf{Coll}}(s)\mathbf{IM}$. Considering the collateral already received by the lender, the variation between two margin calls in times q and s (we recall the margin call occur on each business day), is:

$$\Delta C^l(s)=[S(s)\mathbf{IM}-V_{\mathbf{Coll}}(s)\mathbf{IM}]-C^l(q) \quad (42)$$

Negative variations and values means the buyer posts instead of receiving collateral. We can set the initial collateral at the inception of the contract in $t=0$ equal to:

$$C^l(t)=V_{\mathbf{Coll}}(t)\mathbf{IM}=S(t)\mathbf{IM}$$

4.4 Valuation

A security lending contract is valued according to the principles recalled above and stated in Castagna [6]: so we need to consider the **CVA** and the **FVA** of the evaluating party, assuming that its default is not triggered by the counterparty's default.

Valuation for the Security Lender

We consider a contract with a fixed expiry in T (an additional complexity is given by the possibility that the contract is open-ended). The value of the contract to lender is:

$$\mathbf{SecLending}(t,T) = \mathbf{E}^Q\left[\sum_{i=0}^{N} \mathcal{D}(t,T_i)S(T_0)R\tau(T_{i-1},T_i)\right] - \mathbf{CVA}^l(t,T) + \mathbf{LVA}^l(t,T) \quad (43)$$

$\tau(T_{i-1},T_i)$ is the day count fraction (according to the convention chosen in the contract), between two payment dates (T_{i-1} and T_i, with $T_0 = t$ and $T_N = T$), and it is multiplied by the contract fee rate R, applied to the value of the loan at the start $S(T_0)$. The $\mathbf{LVA}^l$ is defined as in formula (19): this component exists only when collateral is paid in cash and the lender has the opportunity to reinvest it at the risk-free rate (on a risk-adjusted basis), paying an interest to the borrower.

The **CVA**, from the lender point of view, of the security lending at time t, assuming a time horizon T; this is defined similarly to what we have seen for a repo contract. Formally, we have:

$$\mathbf{CVA}^l(t,T) = \mathbf{E}^Q\left[\int_t^T \mathcal{D}(t,u)\max\left[S(u) - C^l(u), 0\right] d\mathbf{PD}_d(t,u)\right] \quad (44)$$

Taking into account the initial margin and the daily collateral maintenance, the $\mathbf{CVA}^l$ is quite negligible.

Hence, since the $\mathbf{CVA}^l(t,T)$ can be assumed to be zero for practical purposes, if the collateral is posted in securities, so that also $\mathbf{LVA}^l = 0$, then the contract has always a positive value at the inception for the lender, which seems an arbitrage at a first look. What are missing here is the fact that the security can be lent because it was purchased some time before the start of the contract, and it is unencumbered: it has not been repoed out to finance its own purchase or pledged anyway. The purchase is then funded unsecured, over a time horizon that has been chosen by the lender given the original intent; for example, it can be a security held for liquidity buffer purposes, or it can be held for investment reasons. So it can be expected to be funded for a longer or shorter period, but in any case the lender will pay an unsecured funding spread.[16] Let s^l be the funding spread paid by the lender over the risk-free rate, assumed a constant for simplicity's sake. For the period corresponding to the duration of the security lending contract, the present value in t of the cost paid to fund the security is:

$$\mathbf{FVA}^S(t,T) = \mathbf{E}^Q\left[\int_t^T \mathcal{D}(t,u)s^l S(t_{Buy})du\right] \quad (45)$$

[16]It is out of the scope of this paper to model the unsecured funding spread. A possible approach to account for the funding mix available to the lender, and the related costs and risks, is outlined in chapter 11 of Castagna and Fede [7].

where $S(t_{Buy})$ is the cash outflow the lender funded when purchased the security in time $t_{Buy} \leq t$ (we have not considered additional cash flows possibly occurring in the period). The funding costs should have been already taken into account when evaluating the economic results of the original transaction: for example, if the security was bought for investment reasons, the cost of carry (*i.e.*: the costs to fund its purchase) should be deducted from the return generated. Now, if the security is lent out for a given period, then the lender can earn a fee that can be used to abate the funding costs. Let ρ by the fraction of funding costs the lender aims at saving when dealing a contract; the valuation equation (43) modifies by including also this component as follows:

$$\begin{aligned}\mathbf{SecLending}(t,T) =& \mathbf{E}^Q\left[\sum_{i=0}^{N}\mathcal{D}(t,T_i)S(T_0)R\tau(T_{i-1},T_i)\right] - \mathbf{CVA}^l(t,T)\\ &- \mathbf{FVA}^l(t,T) + \mathbf{LVA}^l(t,T)\end{aligned} \tag{46}$$

where $\mathbf{FVA}^l(t,T) = \rho\mathbf{FVA}^S(t,T)$ is the fraction of the (present value of) funding costs to charge on the security lending contract. Letting the **CVA** and the **LVA** be both equal to zero, the fair lending fee R is the one making nil the value of the contract at inception:

$$R = \frac{\mathbf{FVA}^l(t,T)}{\sum_{i=0}^{N} P^D(t,T_i)S(T_0)\tau(T_{i-1},T_i)} \tag{47}$$

The valuation problem is now shifted to the choice of the parameter ρ. We here offer only at a very short discussion, hinting at very general ideas: when setting up a security lending business, the lender must decide how much of the funding costs (attributable to the purchase of the securities) have to be recovered via the security lending activity. This percentage can be set based on considerations completely unrelated to the market demand to borrow securities, or it can totally depend on it, so that the lender accepts any fee determined by the market demand and supply forces. So ρ can be a free and independent choice, or it can be simply deduced from the market prevailing fees, so that the value of the security lending contracts are always fair to the lender. In the end, the valuation of the security lending contracts can be quite flexible, and arbitrary somehow.

Valuation for the Security Borrower

The valuation to the borrower is operated similarly as before; the value is:

$$\mathbf{SecLending}(t,T) = \mathbf{E}^Q\left[-\sum_{i=0}^{N}\mathcal{D}(t,T_i)S(T_0)R\tau(T_{i-1},T_i)\right] - \mathbf{CVA}^b(t,T) + \mathbf{LVA}^b(t,T) \tag{48}$$

which is simply the present value of the stream of fees paid, considering also the **CVA** and the **LVA** from the borrower perspective. Neglecting once again the **LVA**, as if collateral were never posted in cash, we need to focus on the **CVA**: it can be computed after noting that the exposure of the borrower to the lender is given by the difference between the values of the collateral posted and of the security. In formula we have:

$$\mathbf{CVA}^b(t,T) = \mathbf{E}^Q\left[\int_t^T \mathcal{D}(t,u)\max\left[C^b(u) - S(u), 0\right] d\mathbf{PD}_l(t,u)\right] \tag{49}$$

where $\mathbf{PD}_l$ is the default probability of the security lender and C^b is the collateral received by the security borrower ($C^b = -C^l$). It is worth noting that, if the one hand the $\mathbf{CVA}^l$ is made negligible, if anything, by the initial margin and the collateral maintenance mechanism, on the other hand these two factors make the $\mathbf{CVA}^b(t,T)$ surely a positive quantity (*i.e.*: an expected loss on the lender's default always greater than zero).

We do not take into account in (48) any **FVA**, even though strictly speaking a small cost is paid by the borrower to fund the negative cash flows related to the fee payments.

Thus, summing up the results above, the value of the contract to the borrower is always negative, which obviously dos not imply any arbitrage opportunity. The negative value should be considered a cost the borrower has to pay to borrow the security, similarly to the costs paid borrow money. They are factored in the total P&L of the borrower when computing the economic result of its business activity.

5 Conclusion

This paper is a summary of the current legal frameworks and best practices adopted in the trading of contracts that are the main market instruments to manage and transform the collateral within a financial institution. We have dwelt also on valuation aspects that are crucial to assess the total economic results originated by trading in these instruments. More complex contracts and sophisticated trading strategies, based on the investigated instruments, will be the object of future research.

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