Iason ltd. and Energisk.org are the editors of the scientific material published in this newsletter. Iason is the publisher. You may not reproduce or transmit any part of this document in any form or by any means, electronic or mechanical, including photocopying and recording, for any purpose without the express written permission of Iason ltd. Both editors are not responsible for any consequence directly or indirectly stemming from the use of any kind of adoption of the methods, models, and ideas appearing in the contributions contained in the issues, nor they assume any responsibility related to the appropriateness and/or truth of numbers, figures, and statements expressed by the authors of those contributions.
Table of Contents

Editorial pag. 3

banking & finance

Sight Deposit and Non-Maturing Liability Modelling pag. 7
Antonio Castagna and Francesco Manenti

Dividend Risk and Dividend-based Instruments pag. 19
Luca Olivo

Analytical Credit VaR under Different Scenarios for Probabilities of Default and Recoveries pag. 25
Antonio Castagna, Fabio Mercurio, and Paola Mosconi

Optimal Quantization Methods pag. 35
Gaetano Marino

energy & commodity finance

Asian Options with Jumps pag. 47
Marina Marena, Andrea Roncoroni, and Gianluca Fusai

A General Approach to Modelling and Pricing in Energy and Weather Markets pag. 57
Fred Espen Benth

Trading Oil Spreads: Statistical Arbitrage pag. 61
Mark Cummins and Andrea Bucca

crash course

Monetary Measurement of Risk: A Critical Overview pag. 67
Lionel Lecesne and Andrea Roncoroni

Front Cover: Umberto Boccioni Dinamismo di un Ciclista, 1913.
Dear Readers,

we are delighted to announce the publication of the inaugural issue of Argo – New frontiers in practical risk management. Argo is a new digital quarterly magazine born from the joint effort of two specialists in risk management. Iason is a consulting company operating in the fields of advanced quantitative modeling and financial derivative pricing for the banking industry. Energisk.org is a major research and consulting network acting in the field of energy and commodity finance.

Argo addresses a number of practically useful and intellectually stimulating contributions to the wide audience of market practitioners and academic scholars concerned with financial and corporate risk management issues. A focus is put on the sectors of banking, financial services, and industrial corporations, including producers and end consumers.

Argo aims at investigating cutting-edge issues in risk management using a fair balance between theoretical soundness and concrete pragmatism. An approach that is reminiscent of a famous sentence made by the German philosopher Immanuel Kant, who said that “experience without theory is blind, but theory without experience is mere intellectual play”.

At first glance, Argo may appear to be a scientific magazine: we believe that it actually is more than that. Not only do published articles include deeply formal treatment of the subjects under investigation, at least whenever their complexity calls for that; but also they highlight the practical extent of the proposed methodologies, techniques, and general items using a case-study and multiple-example approach.

Argo publishes a few articles in connection to current and past R&D and consulting activity carried over by the sponsoring institutions. One may easily detect these contributions through a tag reporting either Iason or Energisk.org logo on the heading page of the concerned article.
Yet, *Argo* is more than just a company’s newsletter. Research work encompasses also contributions from academics and practitioners from a variety of institutions and companies all over the world. Argo thus provides the reader with an effective tool to follow development, uses, and critics of current methodologies, models, and market practices in real-time.

The scrutiny process of the articles is coordinated by a scientific committee including Antonio Castagna and Prof. Gianluca Fusai: they will vouch for the scientific accuracy of the published material. The magazine is organized in two main sections: Section I addresses Banking & Finance issues, while Section II focusses on Energy & Commodities finance. Other contents consist of special interviews and crash courses always related to financial markets and energy fields.

In this first issue you can read about an innovative financial model to manage the risk of sight deposits and non-maturing liabilities, in the article by Antonio Castagna and Francesco Manenti; an implementation of hedging strategies with dividend instruments in the article by Luca Olivo; the application of optimal quantization methods in option pricing in the interesting contribution by Gaetano Marino; finally, an analytical computation for the credit VaR is presented in the article by Antonio Castagna, Fabio Mercurio and Paola Mosconi. This will conclude the Banking & Finance section.

In the Energy & Commodities finance section, Fred Espen Berth (University of Oslo) introduces the reader with a general approach to modelling and pricing weather-linked claims; Mark Cummins (Dublin City University) and Andrea Bucca (Glencore) test the profitability of certain statistical arbitrage strategies on crude oil spreads. Finally, Lionel Lecesne (University of Cergy) and Andrea Roncoroni (ESSEC Business School) put forward the first part of critical overview on monetary measurement of risk, a subject of particular interest for practitioners in financial and industrial sectors concerned with financial risk.

We hope you will appreciate the final result and you will be willing to contribute to the future issues by submitting your works. Submission policies are shown in the magazine.

Enjoy reading *Argo*!

Antonio Castagna
Andrea Roncoroni
Luca Olivo
Banking & Finance

Asset & Liability Management

Derivative Pricing

Credit Risk

Quantitative Finance
Iason Ltd provides solutions that enable managers, risk-control officers and front office people to understand and value their portfolios. All solutions are built on the principles of soundness, accuracy and clearness.

**Pricing**

**Modern Derivatives Pricing**
Sound models to value derivative instruments according with the best and newest market practice (CVA, DVA and liquidity).

**Complex Products**
Sophisticated analytical models, supported by research, in order to effectively and efficiently evaluate all kind of exotic structures, accounting for all the relevant risk factors.

**Revaluation of Books**
The service is for those institutions not interested in building a trading and risk management system, but simply needing a fair revaluation of their (possibly very complex) derivatives positions.

**Risk Measurement**

**Credit Risk**
Innovative approach to Credit VaR (CreVar) modeling and computation.

**Market Risk**
Innovative and sophisticated solutions for market risk measure and management.

**Counterparty Risk**
Pioneers in the development of CCR and CVA measurement models and applications with devoted internal teams in many major banks.

**ALM and Liquidity Risk**
Consulting services to support ALM and treasury department of financial institutions. Innovative models taking into account the interrelation between funding, liquidity and CCR.

To learn more: www.iasonltd.com.
To contact us: info@iasonltd.com.
Follow us on:

- @iason_ltd
- company web page iason
- iason.network
Sight Deposit and Non-Maturing Liability Modelling

In this article we present a review of the most significant approaches provided by the literature and the market practice for the modeling of non-maturing deposits accounts. We describe the bond portfolio replication approach and then move to the class of stochastic factor models, showing how the latter are capable of provide more effective tools for the interest rate and liquidity risk management of these balance-sheet items.

Antonio CASTAGNA
Francesco MANENTI

The modelling of deposits and non-maturing liabilities is a crucial task for the liquidity management of a financial institution. It has become even much more crucial in the current environment after the liquidity crisis that affected the money market in 2008/2009. Typically ALM departments of banks, involved in the management of interest rate and liquidity risks, face the task of forecasting deposit volumes, so as to design and implement consequent liquidity strategies. Moreover deposit accounts represent the main source of funding for the bank, primarily for those institutions focused on the retail business, and they heavily contribute to the funding available in every period for lending activity. Amongst the different funding sources, deposits have lower costs, so that in a funding mix they contribute to abate the total cost of funding. Deposit contracts indeed have the peculiar feature of not-having a predetermined maturity, since the holder is free to withdraw the whole amount in every time. The liquidity risk for the bank thus arises from the mismatch between the term structures of assets and liabilities of the bank’s balance sheet, being the liabilities mostly made up by non-maturing items and the asset by long term investments (such as mortgage loans).

The optionality embedded in non-maturing products, related to the possibility for the customer to arbitrarily choose any desired schedule of principal cash-flows, has to be understood and accounted for when performing liabilities valuation and hedging the market and liquidity risk. Thus a sound model is essential to deal with nested optionality for liquidity risk management purposes.

Modelling Approaches

There are two different approaches in the financial literature and the market practice for the modelling of deposits’ balance evolution:

- Bond portfolio replication,
- OAS models.

The bond portfolio replication, probably the most common approach adopted by banks, can be shortly described as follows.

First, the total deposits’ amount is split in two components:

- a core part that is assumed to be not sensible to market variable evolution, such as interest rates and deposit rates. This fraction of the total volume of deposits is supposed to decline gradually on a medium-long term period (say, 10 or 15 years) and to amortise completely at the end of it.
- a volatile part that is assumed to be withdrawn by depositors over a short horizon. This frac-
tion basically refers to the component of the total volume of deposits that is normally used by depositors to match their liquidity needs.

Secondly, the core part is hedged with a portfolio of vanilla bonds and money market instruments, whose weights are computed by solving an optimisation problem that could be set according to different rules. Typically, the portfolio weights are chosen so as to replicate the amortisation schedule of deposits or, equivalently said, their duration. In this way the replication portfolio tries and preserve the economic value of the deposits (as defined later on) against the market interest rates’ movements. Another constraint, usually imposed in the choice of portfolio weights, is the target return expressed as a certain margin over the market rates. Since deposit rates are updated, within a relatively large freedom of action, by banks to align them to market rates, the replication portfolio can comprise fixed rate bonds, to match the inelastic part of the deposit rates that is not reactive to changes of market rates, and floating rate bonds, to match the elastic part of the deposits rates. The process to re-balance the bond portfolio, although simple in theory, is quite convoluted in practice. For an more detailed explanation of the mechanism, see Bardenhewer in [8].

Thirdly, the volatile part is invested in very short term assets, typically in overnight deposits, and it represents a liquidity buffer to cope with the daily withdrawals by depositors. The critical point of this approach stands in the estimation of the amortisation schedule of non-maturing accounts, that is performed on statistical bases and have to be reconsidered periodically. One of the flaws of the bond replica approach is that the risk-factors affecting the evolution of the deposits are not modelled as stochastic variables. As such, once the statistical analysis is performed, the weights are applied by considering the current market value of the relevant factors (basically, market and deposit rates) without considering their future evolution. This flaw is removed, at least partially, by the so called Option Adjusted Spread (OAS) approach, which we prefer to define as Stochastic Factor (SF) approach. The approach is not in principle different from the Bond Portfolio Replica approach: one tries and identify statistically how the evolution of the deposit’s volume is linked to risk factors (typically market and interest rates) and then set up a hedge portfolio that covers the exposure.

The main difference lies in that, differently from the Bond Portfolio Replica, in the SF approach the weights of the hedging instruments are computed considering the future random evolution of the risk factors, so that the hedging activity resembles the dynamic replication of derivatives contracts. The hedging portfolio is revised based on the market movements of the risk factors, depending on the stochastic process adopted to model them. We prefer to work with a SF approach to model deposit volumes for several reasons. First, we think that the SF approach is more advanced under a modelling perspective, taking into account explicitly the stochastic nature of the risk factors. Secondly, if the Bond Portfolio Replica can be deemed adequate for hedging the interest rate margin and the economic value of the deposits, under a liquidity risk management point of view the SF approach is superior, for the very fact that it is possible to jointly evaluate within a unified consistent framework the effects of the risk factors both on the economic value and on the future inflows and outflows of the deposits. Thirdly, it is easier to include in the SF approach complex behavioural functions linking the evolution of the volumes to the risk-factors. Finally, bank-run events can be also considered and properly taken into account in SF approach, whereas it seems quite difficult their inclusion within a Bond Portfolio Replica approach.

The Stochastic Factor Approach

The first attempt to apply the SF approach, within an arbitrage-free derivatives pricing framework, to deposit accounts was made by Jarrow and van Deventer [2]. They derived a valuation framework for deposits based on the analogy between these liabilities and an exotic swap whose principal depends on the past history of market rates. They provide a linear specification for deposit volumes evolution applied to U.S. federal data.

Other similar models have been proposed, within the SF approach: it is possible to identify three building blocks common to all of them:

1. a stochastic process for the interest rates: in the above mentioned Jarrow and van Deventer, for example, it is the Vasicek model;
2. a stochastic model for the deposit rates: typically these are linked to the interest rates by

---

1We think that OAS is misleading for a number of reasons: the approach does not explicitly model any optionality and does not adjust any spread, as it will be clear from what we will show below. The name is likely derived from a contagion from the (bad) practice, in the fixed income market, to use an effective discount rate to price assets taking into account embedded optionals (whence the name).

2See, amongst others, Frauendorfer and Schürle in [2], Dewachter et al., Kalkbrener and Willing [5] and Blöchlinger [4]
means of a more or less complex function;

3. a model for the evolution of the volume of deposits: since this is linked by some functional forms to the two risk factors at points 1 and 2, it is a stochastic process as well.

The specification of deposit volumes dynamics is the crucial feature distinguishing the different SF models: looking things under a micro-economic perspective, volumes depend on the liquidity preference and risk-aversion of depositors, whose behaviour is driven by the opportunity costs between alternative allocations. When market rates rise, depositors have a greater convenience to withdraw money from sight deposits and invest it in other assets offered in the market. SF models can be defined behavioural in the sense that they try to capture the dynamics of depositors’s behaviour with respect to market rates and deposit rates movements. In doing this, these models exploit the option pricing technology, developed since 1970s, and depend on stochastic variables, in contrast with the previously mentioned class on simpler statistical models, as mentioned above. Depositors’ behaviour is synthesized in a behavioural function that depends on risk factors and determines their choice in terms of amount allocated in deposits. This function could be specified in various forms, allowing for different degrees of complexity. Given their stochastic nature, those models are suitable to be implemented in simulation-based framework like Monte Carlo methods. Since closed-form formulae for deposits’ value are expressed as risk-neutral expectations, the scenario generation process has to be accomplished with respect to the equivalent martingale probability measure. For liquidity management purposes, it is more appropriate to use real-world parameter processes. In what follows we will not make any difference, though: assuming a risk-aversion parameter equal to zero, real-world process for interest rates clash with risk-neutral ones. We propose a specification of the SF approach that we think is enough parsimonious, yet effective.

**Modelling of Market Interest Rates**

The dynamics for the market interest rates can be chosen rather arbitrarily in the class of short rate models. In the our specification we adopted a one-factor CIR++ model: we know that such model is capable to to perfectly match the current observed term structure of risk-free zero rates. The market instantaneous risk-free rate is thus given by

\[ r_t = x_t + \phi_t \]

where \( x_t \) has dynamics

\[ dx_t = k(\theta - x_t)\, dt + \sigma \sqrt{x_t}\, dW_t \]

and \( \phi_t \) is a deterministic function of time.

**Modelling of Deposit Rates**

The deposit rate evolution is linked to the pricing policy of the banks, providing a tool that can be exploited to drive deposits volume across time. It is reasonable to think that an increase of the deposit rate will work as an incentive for existing depositors not to withdraw from their accounts or to even increase the amount deposited. The rate paid by the bank on deposit accounts can be determined according to different rules. Here are some examples:

1. constant spread below market rates:
   \[ d_t = \max(r_t - \alpha, 0) \]
   to avoid having negative rates on the deposit, there is a floor at zero.

2. a proportion \( \alpha \) of market rates:
   \[ d_t = \alpha r_t \]

3. a function similar to the two above but dependent also on the amount deposited:
   \[ d_t = \sum_{j=1}^{m} i_j(r_t) \mathbf{1}_{\{D_r, D_{r+1}\}} D_t \]

where \( D \) and \( D_{r+1} \) are range of the deposit volume \( D \) producing different level of the deposit’s rate.

We adopt a rule slightly more general than the proportional one, i.e.: a linear affine relation between the deposit rate and the market short rate

\[ d_t = \alpha + \beta r_t + u_t \]  \hspace{1cm} (1)

where \( E(u_t) = 0 \) \( \forall t \). As it will be manifest in what follows, the deposit volume’s evolution depends on the deposit rate, so in this framework the pricing policy function, that is obviously discretionary for the bank, represents a tool to drive deposit volumes and consequently it can be used to define liquidity strategies.

**Modelling of Deposit Volumes: Linear Behavioural Functions**

We can model the evolution of the total deposit volume by establishing a linear relationship between its log-variations and the risk factors, i.e.: the market interest and deposit rates: this is the simplest behavioural functional form we can devise. Moreover we add an autoregressive component, by imposing...
that the log-variation of the volume at a given time is linked to the log-variation of the previous period with a given factor and finally we include also a relationship with the time, so as to detect time-trends. The volume’s evolution is in this case given by the equation:

\[ \log D_t = \gamma_0 + \gamma_1 \log D_{t-1} + \gamma_2 t + \gamma_3 \Delta r_t + \gamma_4 \Delta d_t + \epsilon_t \]  

with \( \Delta \) being the first-order difference operator and \( \epsilon_t \) the idiosyncratic error term with zero mean. This formula is in practice the same as in Jarrow and van Deventer.

The model in 2 is convenient because parameters can be easily estimated on historical data via standard OLS algorithm. The presence of a time component in equation 2 is justified by empirical evidence on deposit series, that exhibit a trend component. This factor could be modelled in alternative ways, substituting the linear trend with a quadratic or exponential one. For interest rate risk management purposes, we can however be interested in understanding how deposit evolution explained only by market and deposit rates’ movements. To this end, we can introduce a reduced version of the model that is estimated excluding the trend component, i.e.:

\[ \log D_t = \gamma_0 + \gamma_1 \log D_{t-1} + \gamma_3 \Delta r_t + \gamma_4 \Delta d_t \]  

Empirical analysis of both the model’s form will be presented below.

Modelling of Deposit Volumes: Non-Linear Behavioural Models

The behavioural function linking the evolutions of deposits’ volume to the risk factors can be also non-linear, possibly involving complex forms. In recent years some efforts have been made to formulate this relation according to more sophisticated functions, trying to describe peculiar features of deposits’ dynamics. The main contribution in this direction was provided by Nyström [6], who introduced in the valuation SF framework we are discussing a non-linear dependency of the deposit volumes dynamics from the interest rates. The formalization of such dynamical behaviour is not trivial and we propose a model specification, inspired to the cited Nyström’s work. The main reason why non-linear behavioural functions have been proposed is a drawback of equation 2: it does not allow to fully capture the empirically observed depositors’ reactions to market and deposit rates’ movements. Actual behaviour exhibits high non-linearity with respect to these, in the sense that it depends not only on variations, as implied by equation 2, but also on the levels of market and deposit rates.

The main idea in modelling the non-linear behaviour is based on the micro-economic liquidity preference theory: depositors (and, generally speaking, investors) prefer to keep their investments in liquidity for low levels of market rates. As market rates increase, the preference for liquidity is counterbalanced so that depositors transfer higher fractions of their income and wealth to less liquid investments. In more detail, the first variable to consider is the total depositors’ income, \( I \), growing at an annual rate \( \rho \) on an aggregated base we could see it as the growth rate of the economy (GDP) or simply the growth rate of the income for each depositor (customer). Secondly, the allocation of the income between deposits and other (less liquid) investments hinges on the following assumptions:

- each depositor modifies his balance on the deposit account targeting a given fraction \( \bar{\lambda} \) of their income \( I \). This level can be interpreted as the amount they need to cover his short-time liquidity needs. At any time \( t \), given the current fraction \( \lambda_t \) of the income invested in deposit, the adjustment toward the target \( \bar{\lambda} \) occurs at a speed \( \xi \);

- there is an interest rate’s strike level \( E \) specific to the customer, such that, when the market rate is above it, then they reconsider the target level and redirects a higher amount to other investments, by a fraction \( \gamma \) of their income;

- there is an deposit rate’s strike level \( F \), specific to the customer, such that, when the rate received on deposits is above it, then they are more reluctant to withdraw money by a fraction \( \delta \) of their income;

Under these assumptions, the evolution of the fraction \( \lambda_t \) of the income allocated in sight deposit is:

\[ \lambda_{t+\Delta t} - \lambda_t = \xi (\bar{\lambda} - \lambda_t) \Delta t + \gamma \mathbf{1}_{(E,\infty)} (r_t) + \delta \mathbf{1}_{(F,\infty)} (d_t) \]  

where \( \mathbf{1}_{(E,\infty)} \) is the indicator function equal to 1 when the condition in the subscript is verified. The income \( I \) grows as follows:

\[ I_{t+\Delta t} - I_t = I_t \rho \Delta t \]  

and the deposit volume at time \( t \) is:

\[ D_t = \lambda_t I_t \]  

In reality, since each depositor has different level of strike rates \( E \) and \( F \), due to their preferences for liquidity, on an aggregated basis, considering all bank’s customers, there is a distribution of strike
rates, reflecting their heterogeneity in behaviour. So, when we pass from the evolution of the single deposit, to the evolution of the total volume of deposits on bank’s balance sheet, strike rates can be thought to be distributed according to any suitable probability function \( h(x) \): in the specification we present here we choose a Gamma function, i.e.:

\[
h(x; \alpha, \beta) = \frac{(x/\beta)^{\alpha-1} \exp(-x/\beta)}{\beta \Gamma(\alpha)}
\]

EXAMPLE 1 The Gamma function is very flexible and it allows for a wide range of possible shapes of the distribution. If we set \( \alpha = 1.5 \) and \( \beta = 0.05 \), or example, we have a distribution labeled as “1” in Figure 1. If \( \alpha = 30 \) and \( \beta = 0.002 \) we have a distribution “2”. It is possible to model the aggregated customers’ behaviour, making it more or less concentrated around specific levels.

We can alternatively use the equivalent functional form of the Gamma distribution written as:

\[
h(x; k, \theta) := \frac{1}{\theta^k \Gamma(k)} e^{-x/\theta} k^{k-1}
\]

This is actually what we will use in the estimation of the parameters from historical data we will show below.

The evolution of total volume of deposits can be written by modifying equation 4 and considering the distributions of the strike rates instead of the single strike rates for each depositor:

\[
\lambda_{t+\Delta t} - \lambda_t = \zeta(\lambda - \lambda_t)\Delta t + \gamma H(r_t, k_1, \theta_1) + \delta H(d_t, k_2, \theta_2)
\]

where \( H(x, k, \theta) = \int_0^x h(u; k, \theta) \) is the Gamma cumulative distribution function.

To make the econometric estimation of the parameters easier, we rewrite equation 7 in the following way:

\[
\lambda_t = \alpha + \beta \lambda_{t-1} + \gamma H(r_t, k_1, \theta_1) + \delta H(d_t, k_2, \theta_2)
\]

where \( \alpha = \zeta \Delta t \) and \( \beta = \zeta \Delta t \). Equation 8 can be applied by the bank to the “average customer”. Given the heterogeneity of the behaviours, given current market and deposit rates, the incentive to change the income allocation by increasing less liquid investments, balanced by the incentive to keep the investment in deposits provided by the deposit rates, is synthesized in the Gamma distribution functions, so that \( H(x, k, \theta) \) turns out to be the cumulative density of the average customer’s strike.

### Economic Evaluation and Risk Management of Deposits

The three building blocks to model the deposits can be used to compute the economic value to the bank of the total amount held on balance sheet. At time \( t = 0 \), for a time horizon \( T \), the economic value is the expected margin that can be earned by the bank on the present and future volume of deposits. In fact, the amount of funds raised by the bank in form of deposits can be invested at in short expiry risk-free investments yielding \( r_i \); on the other hand the deposits cost to the bank the rate \( d_t \) that it has to pay to depositors. In formula:

\[
V^D(0, T) = \sum_{j=1}^n \int_0^T E^Q \left[ (r_i - d_{j,t}) D_{i,t} P^D(0, t) \right] dt
\]

where \( D_{i,t} \) is the amount deposited in the account \( j \) at time \( t \) and \( n \) is the number of deposit accounts. The expectation is taken under the equivalent martingale risk-neutral measure \( Q \). Equation 9 is the expected Net Interest Margin to the bank over the period \( [0, T] \), for all the deposits accounts, discounted in 0 by the risk-free discount factor \( p^D \). As suggested by Jarrow and van Deventer [2], the value of
deposits can be seen as the value of an exotic swap, paying the floating rate $d_{j,t}$ and receiving the floating rate $r_t$, on the stochastic principal $D_{j,t}$ for the period between 0 and $T$. The approach we outlined above is also a good tool for liquidity risk management, since it can be used to predict expected or stressed (at a given confidence level) evolution of deposit volume. To compute these metrics, we need to launch a Monte Carlo simulation on the two risk factors, i.e. the risk-free instantaneous interest rate and deposit rate. We operate the following steps:

- given a time-horizon $T$, divide the period $[0, T]$ in $M$ steps;
- simulate $N$ paths for each risk factor;
- compute the expected level of deposit volume $V(0,T_i)$ at each step $i \in \{0, 1, ..., M\}$, by averaging out on the $N$ scenarios, by means of equation 2 or 8:
  \[
  D^e(T_i) = E[D(T_i)] = \frac{\sum_{m=1}^{M} D^m(T_i)}{M}
  \]
- compute the stressed level of deposit volume at a given confidence level $p$, $V^p(0,T_i)$ at each step at each step $i \in \{0, 1, ..., M\}$, based in the $M$ scenarios. Since for liquidity risk management purposes the bank is interested at the minimum levels of the deposit volume at a given time $T_i$, then we define the stressed level at $p$ confidence level as:
  \[
  D^p(T_i) = \inf \{D(T_i) : \Pr[D(T_i) < D^p(T_i)] \geq p\}
  \]

Banks can be interested in computing the minimum level of deposits during the entire period included between the reference time (say, 0) and a given time $T_i$: this is actually the value that corresponds to the actual available liquidity that can be used for investments expiring in $T_i$. To this end it is useful to introduce the process of the minima of the deposit volume, defined as:

\[
D^{\min}(T_i) = \min_{0 \leq s \leq T_i} D(s)
\]

Basically the process exclude all the growth of the volume of deposits due to new deposits or to an increase of the amount of the exiting ones, but it considers only the abating effects that the risk factors produce. The metric is also consistent with the factual truth that in any case the bank can never invest more than the existing amount of deposits it has on its balance sheet. The SF approach can be used also for interest rate management purposes. Once we have computed the economic value of deposits, it is straightforward to compute its sensitivities to risk-factors to set up hedging strategies with liquid market instruments such as swaps. To
this end, we can calculate the sensitivities of deposits’ economic value to perturbations in the market zero-rate curve. The sensitivity to the forward rate \( F(0; t_i, t_{i+1}) = F_i(0) \) is obtained numerically by means of the following:

\[
\Delta V(0, T; F_i(0)) = V(0, T; \hat{F}_i(0)) - V(0, T; F_i(0))
\]

where \( V(\cdot) \) is provided by (9) and \( \hat{F}_i(0) \) is the relevant forward rate bumped by a given amount (e.g.: 10 bps). We have assumed that the instantaneous short rate follows a one-factor CIR++ dynamics. Assuming now that the initial zero-rate curve generated by the model, i.e.: the series \( \{P^D(0, T_i)\}_{i=1}^{n} \) perfectly matches the market observed term structure, we have to modify the short rate dynamics in a way that produces the desired bump on the forward rates time 0, by suitably modifying the deterministic time dependent term \( \phi(t) \) of the CIR++ process. This is easily done: let \( bmp \) be the size of the bump to the term structure of starting forward rate \( F_i(0) \); in the CIR++ the tilted forward \( \hat{F}_i(0) \) is obtained by modifying the integrated time dependent function \( \phi(t) \) as:

\[
\int_{T_i}^{T_{i+1}} \phi(s)ds \rightarrow \int_{T_i}^{T_{i+1}} \phi(s) + \frac{\ln(bmp)}{\tau_i}ds
\]

where \( \tau_i = T_{i+1} - T_i \). We present below some practical applications to the approach sketched above.

EXAMPLE 2

We perform an empirical estimation and test of the the SF approach, with the two behavioural functions we have presented above, based on public aggregated data for sight deposits in Italy. We considered a sample of monthly observations in the period 3/1999 : 4/2012 for sight deposits’ total volume and average deposit rates paid by the bank. Data for deposits are published by Bank of Italy (Bollettino Statistico). We considered the euro 1-month overnight index average (Eonia swap) rate as a proxy for the market short risk-free rate: values for the analysis period are plotted in Figure 2 on the left. The CIR model for the market rate was calibrated on the time series of Eonia rates via Kalman filter, and the resulting values for the parameters are:

\[
\kappa = 0.053, \ \theta = 7.3, \ \sigma = 8.8\%
\]

For the second building block (deposit rates), the linear relation between market rates and deposit rates in Equation 1 has been estimated via standard OLS, results are shown in Table 1. Figure 2 plots on the right the actual time series of deposit rates and fitted values from the estimated regression. The model shows a good fitting of the time series and we can observe that the linear affine relation is strongly consistent with the data.

Finally, we need to adopt a behavioural function. We start with the linear model for deposit volumes in Equation 2: estimation results are shown in Table 2 and also in this case the model proves to be a good explanation of the data. We note that the signs of coefficients multiplying, respectively, the variations in the market rate and the variations in the deposit rate, are opposite as expected. Figure 3 plots actual and fitted time series of deposit volumes. We can now use estimated parameters to compute the economic value of deposits via Monte Carlo simulations of the Formula 9. The standard approach requires to generate a number of simulated paths for the risk factors by means of the estimated dynamics, following these steps:

- compute 10,000 paths for the market rate evolution, simulated with the CIR dynamics;
- for each path, compute the corresponding path for the deposit rate and the deposit volume according to estimated regressions (equations 1 and 2);
- compute deposits value at each time steps in the simulation period;
- sum discounted values path by path and average them to obtain the present value of the total amount of deposits.
**FIGURE 4:** From the top on the left clockwise: simulated paths for 1-month Eonia swap rate, deposit rate and deposit volume, term structure of expected and minimum (99% c.l) future volumes.

**FIGURE 5:** On the left, simulated paths for deposit volume; on the right: term structure of expected and minimum (99% c.l) future volumes.

**FIGURE 6:** On the left: time series of Italian nominal GDP for the sample 3/1999:4/2012; quarterly data are linearly interpolated to obtain the monthly time series. On the right: Actual time series of deposit volumes vs. fitted values for the non-linear behavioural model.
We now estimate the parameters of the non-linear behavioural model in Equation 8, via a Non-Linear Least Squares algorithm; we still use the same dataset as above, i.e. the sample 3/1999 : 4/2012 of monthly data for non-maturing deposits volumes, 1-month Eonia swap rates and deposit rates. In this case, what we actually model is the evolution of the proportion $\lambda$ of the depositor’s income held in a sight deposit. At an aggregated level, we approximated the total income with the nominal GDP, so that the fraction $\lambda$ will be referred to this quantity.Since we are working with Italian deposits, we take the Italian GDP data that are published quarterly and we operate a linear interpolation to obtain monthly values.\footnote{We are aware this is likely not the most sound way to interpolate GDP data, but we think it is reasonably good for the limited purpose of our analysis.} The reconstructed nominal GDP time series, for the estimation period we consider, is shown in Figure 6 on the left. Estimated coefficients and their significance are shown in Table 4. Figure 7 plots the pdf of the strike respectively for the market ($E$) and the deposit ($F$) rates. We can see that the cumulative density functions reach their maximum when the market rate exceeds 3.55% and deposit rate exceeds 4.25%. These should be considered the levels for market interest rates and deposit rates when most of customers consider re-allocating the fraction of income held in deposits on other investments. The regression has an $R^2$ value lower than the linear model tested before: this is also confirmed by the plot of the actual vs fitted deposits’ volumes in Figure 6 on the right. As already done for the linear model, we can compute the economic value of deposits with a Montecarlo simulation. Figure 8 shows simulated paths deposits’ volumes and the term structure of expected and minimum volumes. With a simulation period of 10 years and an initial volume of 834,467 bln Eur, the estimated deposits’ economic value is 88,614, so the non-linear model is more conservative than the linear one. Also for the non-linear model, we can run Montecarlo simulations after freezing the time-trend (which in this case means keeping the GDP constant to the initial level) and the deposit rate. In this way we isolate the effect produced by the market interest rates on the deposits’ volume.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Significance (p-values)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>0.19</td>
</tr>
<tr>
<td>Lagged $D_{t-1}$</td>
<td>0.98</td>
</tr>
<tr>
<td>Market rate’s variations $\Delta r_t$</td>
<td>-4.1</td>
</tr>
<tr>
<td>Deposit rate’s variation $\Delta d_t$</td>
<td>6.52</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.99</td>
</tr>
<tr>
<td>F statistics</td>
<td>5837</td>
</tr>
<tr>
<td>F significance</td>
<td>1.66E-157</td>
</tr>
</tbody>
</table>

**TABLE 3**: Regression results for the reduced version of the linear behavioural equation 3.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Significance (p-values)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>0.25</td>
</tr>
<tr>
<td>Lagged $\lambda_{t-1}$</td>
<td>0.53</td>
</tr>
<tr>
<td>Gamma market rates H($r_t$)</td>
<td>-0.09</td>
</tr>
<tr>
<td>Gamma market rates $\beta_1$</td>
<td>18.77</td>
</tr>
<tr>
<td>Gamma market rates $\beta_2$</td>
<td>0.001</td>
</tr>
<tr>
<td>Gamma market rates $\beta_2$</td>
<td>0.01</td>
</tr>
<tr>
<td>Gamma deposit rates $H(d_t)$</td>
<td>0.14</td>
</tr>
<tr>
<td>Gamma deposit rates $\delta_1$</td>
<td>24.26</td>
</tr>
<tr>
<td>Gamma deposit rates $\delta_2$</td>
<td>0.001</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.97</td>
</tr>
<tr>
<td>F statistics</td>
<td>4518</td>
</tr>
<tr>
<td>F significance</td>
<td>1.1767E-157</td>
</tr>
</tbody>
</table>

**TABLE 4**: Regression results for the non-linear behavioural equation 8.

<table>
<thead>
<tr>
<th>Years</th>
<th>Sensitivity</th>
<th>Sensitivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>730</td>
<td>524</td>
</tr>
<tr>
<td>2</td>
<td>770</td>
<td>683</td>
</tr>
<tr>
<td>3</td>
<td>810</td>
<td>663</td>
</tr>
<tr>
<td>4</td>
<td>860</td>
<td>657</td>
</tr>
<tr>
<td>5</td>
<td>910</td>
<td>659</td>
</tr>
<tr>
<td>6</td>
<td>960</td>
<td>664</td>
</tr>
<tr>
<td>7</td>
<td>1000</td>
<td>678</td>
</tr>
<tr>
<td>8</td>
<td>1050</td>
<td>689</td>
</tr>
<tr>
<td>9</td>
<td>1090</td>
<td>703</td>
</tr>
<tr>
<td>10</td>
<td>1100</td>
<td>723</td>
</tr>
</tbody>
</table>

**TABLE 5**: Sensitivities to 1Y1Y forward Eonia swap rates for 10 bps up to 10 years, for the linear and non-linear model.

The results for the case when only the time-trend is frozen, are shown in Figure 9. It is worth noting that without time-trend, the fraction of income held in deposits rapidly reach the minimum and then, given the autoregressive nature of the model in Equation 8, it keeps con-
A comparison between the linear and non-linear model, as far as the expected and minimum level of deposits’ volume are concerned, is shown in Figure 11. It is quite clear that the non-linear model seems to be much more conservative in terms of expected and minimum level of volumes. We present a comparison also of the market rate sensitivities of the economic value of the deposits obtained by the linear and non-linear model. In Table 5 sensitivities to the 1-year forward (risk-free) Eonia rates, fixed every year up to 10 years, are shown. Sensitivities are referred to a bump of the relevant forward rate of 10 basis points. The linear model has bigger sensitivities due to the higher volumes, and hence higher economic value, expected in the future.

Inclusion of Bank-Runs

It can be interesting to include the possibility of a bank-run in the future, due to a lack of confidence of the depositors in the creditworthiness and the accountability of the bank. If this occurs, it is reasonable to expect a sharp and sudden decline in the deposits’ volumes. To take into account a bank-run, one needs to consider some variable that is linked to the bank’s credit robustness (or the lack of it). One possible solution could be the credit spread of the bank, on short or long-term debt: it can be either extracted from market quotes of the bonds issued by the bank, of from bank’s CDS quotes. As for the model, for the very nature of the bank-run, the non-linear behavioural model is more suitable to accommodate for it. In fact, it is possible to add an additional behavioural function, related to the bank’s credit spread, which will likely be densely concentrated around a high level (denoting an idiosyncratic critical condition). The inclusion of the bank-run is operated by extending formula (8) as follows:

$$\lambda_t = \alpha + \beta \lambda_t + \gamma H(r_t; k_1, \theta_1) + \delta H (i_t; k_2, \theta_2) + \eta H(s^B_t; k_3, \theta_3)$$

(11)

The new behavioural function $H(s^B_t; k_3, \theta_3)$ is still a Gamma function taking as an input the bank’s spread $s^B$. It is quite difficult to estimate the parameters of this function, since it is quite unlikely that the bank experienced many bank-runs. One can resort to bank-runs occurred to comparable banks, but also in this case not to many events can be observed for a robust estimation of the parameters. Nonetheless, the bank can include the bank-run on a judgmental basis, by assigning given values to the behavioural function according to its hypothesis of stressed scenarios.
FIGURE 9: Simulated paths (upper graph) and term structure of expected and minimum (99% c.l) future volumes (lower graph) derived with the estimated non-linear model, when the time-trend is frozen.

FIGURE 10: Simulated paths (upper graph) and term structure of expected and minimum (99% c.l) future volumes (lower graph) derived with the estimated non-linear model, when the time-trend and the deposit rate are frozen.

FIGURE 11: Term structure of expected and minimum (99% c.l) future volumes (lower graph) derived with the linear (equation 2) and the non-linear (equation 8) model.

FIGURE 12: Simulated paths (upper graph) and term structure of expected and minimum (99% c.l) future volumes in the case of bank-run inclusion.
EXAMPLE 3
We extend the non-linear model we estimated in Example to include the possibility of a bank-run. To compute the term structure of expected and minimum volume of deposits, we use Equation (11), with parameters set as shown in Table 4. The parameters of the additional behavioural function are set as follows:

\[ \eta = 0.2, \quad k_3 = 32, \quad \theta_3 = 0.002 \]

Given the parameters of the Gamma function \( k_3 \) and \( \theta_3 \), when the credit spread of the bank reaches a level above 800 bps, then a drop of 20% in the level of the deposits is experienced in each period (we recall we use monthly steps in our examples). To model the credit spread and simulate its evolution in the future, we assume that the default intensity of the bank is given by a CIR process, with parameters:

\[ \lambda_0 = -0.2, \quad \kappa = 0.5, \quad \theta = 5\%, \quad \sigma = 12\% \]

Besides we assume \( a = 60\% \) upon bank’s default. We assume that the spread entering in the behaviour function is the 1-month one, for short-term debt. Figure 12 shows the simulated paths and the term structure of the expected and minimum volume of deposits: when compared with Figure 8 it is evident the lower levels projected by the model.

ABOUT THE AUTHORS
Antonio Castagna is Senior Consultant at Iason.
Email address: antonio.castagna@iasonltd.com
Francesco Manenti is Consultant at Iason and corresponding author. He is currently working on the Risk Management side of a big pan-European bank.
Email address: francesco.manenti@iasonltd.com

ABOUT THE ARTICLE
Submitted: December 2012.
Accepted: February 2013.

REFERENCES

Dividend Risk and Dividend-based Instruments

The article tries to identify dividend risk manifestations in financial markets and its implication in derivative pricing. The presence of such a risk has generated a series of market instruments mainly implied in hedging strategies, but recently also exploited as new sources of risk diversification. Indeed dividend-based instruments may provide good investment opportunities during financial crises, when usually correlations among asset classes tend to surge and performances to worsen.

Luca OLIVO

Kruchen and Vanini[6] probably provide the most effective description of dividend risk: the uncertainty around the possible differences between the (ex post) realized dividend amount and its (ex ante) estimation. Pricing models for equity derivative instruments usually take into account the dividend component, based on market consensus or historical data: in the Black-Scholes model, for example, future and option prices may be affected by the discrete income or the continuous dividend yield of the underlying. Obviously any forecast could be significantly different from its actual realization; the origin of the dividend risk is indeed within this innate uncertainty, that can be described by two main dimensions:

1. **size**: the amount of dividend to be estimated, usually based on analysts consensus or historical data; the estimate can also be extracted from market prices, as an implied level of dividends; of course, the volatility of the estimate increases the more uncertain the consensus or the poorer the availability of market data;

2. **timing**: the ex-dividend date, that becomes extremely relevant when included in the life of the derivative instrument.

As already explained, the manifestation of dividend risk has effects on the pricing of the derivative instruments. Payment of dividends usually generates a Winter in relative stock prices, that indirectly affects derivative value: given the fact that historically dividends represent a significant component of total long-term equity returns, such a manifestation cannot be ignored. Consider for example the S&P500 Index: quite one third of its long-term total return has been due to dividends, thus it would be hard to state dividend risk is irrelevant in pricing issues. Clearly dividend risk directly involves the holders of derivative instruments written on equities stocks or indexes: for example, an investor that goes long on a call option written on a single dividend-paying stock shall be negatively affected to the Winter in spot price if the dividend payment occurs during the option life, since it has no right on the dividend amount. In relation to that, Kruchen and Vanini[6] consider three sets of equity stocks taken from Eurostoxx50, DAX100 and SMI from 1994 to 2006; as derivative instruments, they choose options written on both single stocks and equity indexes. Assuming stochastic discrete cash dividends, they want to measure the impact of dividend risk in derivative prices at different time horizons (6 months, 1 year and 2 years). Their findings are briefly summarized below in order to have an idea about the scope of divided risk in market pricing:

- options written on indexes are significantly less affected by dividend risk than options written on single stocks; it means that the effect of a single dividend cash on option price is more relevant than the one generated by index aggregate dividends;

- the uncertainty around the size of divi-
Dends has quite negligible effect on options with shorter maturities; however, as time-to-maturity increases, volatility around the deviation from actual dividend amount surges, making relevant the impact of divided risk on longer-dated option prices;

- uncertainty around the size of dividends has different effects on European and American options; considering the Europeans, large dividends lead to lower option prices and this effect outpaces the one generated by higher option prices due to lower dividends; for Americans the effect is exactly the contrary;

- uncertainty around the ex-dividend date has equal effect on both American and European options: the effect is significant if the date falls within the time-to-maturity interval; moreover, the nearer the date to maturity, the lower the effect of dividends on option price; furthermore, it is necessary to take into account that, as time-to-maturity decreases, the time value of the option loses importance: so timing effect has to be considered mainly on intrinsic value.

### Dividend Strips

Exploiting some data related to the S&P500 Index, it is possible to effectively identify dividend risk in financial markets. The aim is to find a way in order to extract expectations about future dividends from market data and then compare them with the actual dividend amounts paid out by the index. In doing so let introduce the so-called dividend strips, that shall be defined as the amounts of dividends paid by the index in a certain (future) period of time. A more analytical definition involves the present-value relation to express prices when expected returns are not constant. Suppose there are no arbitrage opportunities in the market and there exists a stochastic discount factor in order to discount single-period cash-flows, then:

\[ P_t = E_t \left[ \sum_{i=1}^{T} M_{t:t+i} D_{t+i} \right] + E_t \left[ \sum_{j=T+1}^{M} t:t+j+D_{t+j+i} \right] \]  

where \( P_t \) is the equity instrument price, \( D_t \) is the discrete amount of dividends to be paid by the instrument and \( M_{t:t+i} \) is the product of stochastic discount factors. From the equation above it is possible to directly derive the **price of a short-term asset** from time \( t \) to time \( T \) :

\[ S_{t,T} = E_t \left[ \sum_{i=1}^{T} M_{t:t+i} D_{t+i} \right] \]  

(2)

Now suppose a dividend strip entitles the owner to all the dividends paid from \( t + T_1 \) to \( t + T_2 \), with \( T_2 > T_1 \); following the present-value approach in eq. (2) the price of a dividend strip at time \( t \) shall be equal to:

\[ S_{t,T_1,T_2} = E_t \left[ \sum_{i=T_1+1}^{T_2} M_{t:t+i} D_{t+i} \right] \]  

(3)

where if \( T_1 = 0 \) and \( T_2 = T \) then **dividend strips can be expressed through the prices of the short-term asset** in (2). These prices are extremely important because represent today expectations about future dividends.

At this point it is possible to replicate the prices of dividend strips synthetically. Eq. (3) can be reformulated exploiting both the future-spot and the put-call parities: since the two methods are alternative, let follow the Binsbergen, Brandt and Koijen[2] approach in exploiting the properties of European options written on the S&P500. Let collect 61 quarterly observations related to the last day of negotiation for each quarter, involving index prices, put and call options; the samples extend from March 1996 to March 20115 and are characterized by four time-to-maturity levels: 3 months (\( \tau = 1 \)), 6 months (\( \tau = 2 \)), 1 year (\( \tau = 4 \)) and 2 years (\( \tau = 8 \)); So eq. (2) shall be reformulated as follows:

\[ S_{t,\tau} = p_{t,\tau} - c_{t,\tau} + P_t - X e^{-i_{t,\tau} \tau} \]  

(4)

with \( \tau = 1, 2, 4, 8 \) quarters. All data are collected from WRDS OptionMetrics database[11] and include put and call option prices at time \( t \) with maturity \( T \) (\( p_{t,\tau} \) and \( c_{t,\tau} \) respectively), strike prices \( X \), index prices \( P_t \) and the continuously compounded risk-free interest rate \( i_{t,\tau} \) (for the period \( \tau = T - t \)). Working that way allows to extract expectations about future dividend growth at different time horizons and then construct several strategies that bet on dividends at different maturities. Let’s take into consideration the changes in log-dividend strip prices (that represent the changes in market expectations) and the actual log-dividend growth in the \( \tau = 4 \) case6.

Table 1 collects the average values and shows that for the entire sample period expectations have strongly underestimated realized dividend growth. However it is interesting to analyze four different

\[ ^5 \text{Before 1996 it is hard to find quoted option data related to all the maturities.} \]

\[ ^6 \text{From now on prices and dividends are expressed in the form of natural logarithms.} \]
phases in the sample period in order to better understand the dynamics throughout; during boom periods (Mar 1997-Jun 2001; Mar 2004-Dec 2007) expectations on future dividend growth have on average overestimated actual dividend realizations; instead after the 9/11 attack (Sep 2001-Dec 2003) and during the recent financial turmoil (Jan 2008-Mar 2011), expectations performed significantly worse than realized amounts. These results are consistent also in other time-to-maturity cases (τ = 1, 2, 8 quarters), confirming that on average for the S&P500 Index the tendency is to underestimate future dividend growth during periods of crisis and overestimate it during boom periods: there are clear manifestations of dividend risk in both directions across the entire period, even if on the long run expectations tend to underestimate realizations. The uncertainty related to the potential mismatch between the changes in $S_{t,T}$ and the changes in actual dividend growth can be effectively managed by one of the most common dividend-based instrument: the dividend swap.

### TABLE 1: Averaged changes in both market-expected and actual dividend growth; quarterly observations from March 1996 to March 2011.

<table>
<thead>
<tr>
<th>Sub-sample</th>
<th>ΔS_{t,A}</th>
<th>Δd_{t,A}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997-2001</td>
<td>5.57</td>
<td>0.58</td>
</tr>
<tr>
<td>2001-2003</td>
<td>-16.80</td>
<td>0.20</td>
</tr>
<tr>
<td>2004-2007</td>
<td>9.14</td>
<td>2.93</td>
</tr>
<tr>
<td>2008-2011</td>
<td>-15.32</td>
<td>-1.21</td>
</tr>
<tr>
<td>1997-2011</td>
<td>-2.25</td>
<td>0.77</td>
</tr>
</tbody>
</table>

Dividend Risk and Dividend Swaps

*Dividend swaps* started to be traded in over-the-counter platforms between 1998 and 1999. In the plain form, they are derivative contracts that commit:

- the *buyer* of the swap to pay to the *seller* the **market-implied level of dividends**, usually multiplied by a notional amount of shares; since this level is synthetically derived from option prices, this can be considered the fixed leg of the swap; the market expectation of dividends under the risk-neutral measure can be referred to the implied strike of the swap.

- the *seller* of the swap to pay to the *buyer* the **realized dividend** that will be detached by the selected underlying\(^8\) at the end of each contract period (always multiplied by the desired exposure per point); this is the floating component of the swap, because market risk makes the actual amount of paid out dividends unpredictable until the expiration.\(^8\)

Basically, the *buyer* of the swap will receive at maturity the actual amount of dividends paid by the underlying between the origination of the contract ($t$) and its expiration ($T$), while the *seller* receives an amount related to the market expectations on those dividends at time $t$; thus the *buyer* bears the **dividend risk**, since at time $t$ you cannot know the exact amount you will receive at maturity. Depending on the chosen underlying, it is possible to distinguish between *index dividend swaps* and *dividend swaps written on single dividend-paying stocks*: while the latters, traded exclusively OTC, are based on the amount of dividends potentially detachable by a single company, the formers are focused on indices with paying-dividend stocks and then consider as underlying the cumulative dividends expressed in index points.

Index dividend swaps have been recently introduced also in regulated exchange platforms, such as Eurex\(^3\), signaling a growing demand especially by institutional investors: these products allow the buyer to take long position on the cumulative gross dividends of the index and the seller to hedge against dividend risk. The choice to quote index dividend swaps instead of single-stock ones may be due to the fact that the uncertainty around payments is strongly reduced: since an index is a basket of stocks from different industrial sectors and dispersion relative to whether the firms will pay dividends or not is surely reduced, the payoff of an index dividend swap can be computed with a higher degree of precision than the payoff of a single-stock dividend swap.

Dividend swaps can be referred as **unfunded** instruments since payments occur only at maturity and no capital investment is required at origination (except an initial margin). In a sense it could be useful to compare them with other **funded** instruments\(^5\) by combining a dividend swap with an investment in a zero-coupon instrument with the same maturity. If a dividend swap that expires in $T$ is traded at the implied strike $K$ at time $t < T$ then you can enter in a long position by investing $Ke^{-h_{t,T}(\tau)}$ now and receiving $K$ at maturity. Since at $T$ you will receive the actual amount of realised dividend $D$ in the period $T-t$ and you will pay the implied strike $K$, combining the investment in the zero-coupon

\(^{3}\)that can be a single stock, a basket of stock or all the components of an index.

\(^{5}\)no cash flows are required at the origination of the transaction, since the payments occur at maturity.

\(^{8}\)such as cash, bond or equities.
instrument with the long dividend swap position you obtain a value of $D$ at maturity. In summary, at time $t$ you pay $Ke^{-it_{\tau}}$ in order to receive $D$ at maturity. Now let focus on dividend swap instruments written on the S&P500 index. Considering the sample of data described in the previous section it is possible to replicate payoffs of dividend swaps at different maturities. Suppose to be a bank that want to hedge its position against dividend risk: one possibility is to sell a dividend swap with underlying the dividends paid out by the S&P500 in a year ($\tau = 4$ quarters). Dividend strip prices in (4) can be considered as the implied strikes of the dividend swaps at different time-to-maturities, while the payments of the floating leg will be determined at maturity by the actual amount of dividends $D_T$ collected by the S&P500 in the period $\tau = T - t$:

\[ S_{t,4} := K \]  

At origination $t < T$ the market-implied expectations about future dividends are given in (5); at maturity $T$ the bank has to pay the actual amount of dividends $D_T$ collected in the period $T - t$ and receive $K$. The payoff by a seller perspective shall be:

\[ K - D_T \]  

where dividend risk is related to the fact that $D_T$ can be significantly different from its expectation at origination. Figure 1 indeed shows the hypothetical development of the payoff in (6) if the swap is continuously rolled-over during the period 1996-2011: in line with results in Table 1, the seller-payoff would be positive during boom periods (when expectations largely overestimated realizations) and negative during financial turmoils.

**Dividend-based instruments as source of risk diversification**

Assume that in general the tendency identified in Table 1 holds in financial markets: it would be possible to construct strategies on dividends that generate profits even in the case of bad market conditions; if an investor decides to buy a swap as the one in eq. (6) during a period of high uncertainty in financial markets, he/she has the possibility to make a profitable deal without incurring in the negative manifestation of dividend risk. This is not such a trivial point, especially if we take into consideration the fact that usually during financial crisis both correlations and volatilities among asset classes tend to surge, generating negative performances.

Let consider the simplest derivative instrument that may be constructed on financial markets: a future contract. Suppose to analytically compute the future price of the short-term asset in (4) as follows:

\[ S_{t,\tau}^f := S_{t,\tau}e^{it_{\tau}} \]  

with $i_t$ the risk-free rate at time $t$; the price in eq. (7) represents the dividend strip future price: since $S_{t,\tau}$ is referred to the short-term asset in (2), a future contract having as underlying such a strategy is not so unrealistic and may be easily traded on markets. The point is to analyze the relationship between the change in dividend strip future prices ($r_{t,\tau}^f$) and the change in S&P500 prices ($r_{t,\tau}^{sp}$) throughout the considered period of time. The linear regression implemented in order to test some degrees of co-movement between the two asset classes has the form in eq. (8):

\[ (r_{t,\tau}^f - i_t) = \alpha_\tau + \beta_\tau (r_{t,\tau}^{sp} - i_t) + \epsilon_{t,\tau}^f \]
TABLE 2: Regression: testing eq. (8) for $\tau = 1, 2, 4, 8$; quarterly data from June 1996 to March 2011. Newey-West HAC standard errors in brackets.

<table>
<thead>
<tr>
<th></th>
<th>$\hat{\beta}_\tau$</th>
<th>st.err.</th>
<th>$t - stat$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau = 1$</td>
<td>-0.2786</td>
<td>0.3901</td>
<td>-0.7141</td>
<td>0.0081</td>
</tr>
<tr>
<td>$\tau = 2$</td>
<td>-0.5170</td>
<td>0.3496</td>
<td>-1.4787</td>
<td>0.0291</td>
</tr>
<tr>
<td>$\tau = 4$</td>
<td>-0.4020</td>
<td>0.3787</td>
<td>-1.0615</td>
<td>0.0164</td>
</tr>
<tr>
<td>$\tau = 8$</td>
<td>-0.3333</td>
<td>0.4919</td>
<td>-0.6776</td>
<td>0.0071</td>
</tr>
</tbody>
</table>

TABLE 3: Correlation coefficients between $r_{\tau}^f$, for $\tau = 1, 2, 4, 8$ and $r_{\tau}^{sp}$ across the entire sample and during the recent financial crisis (2008-2011). Quarterly data from June 1996 to March 2011.

<table>
<thead>
<tr>
<th></th>
<th>$r_{\tau,1}^f$</th>
<th>$r_{\tau,2}^f$</th>
<th>$r_{\tau,4}^f$</th>
<th>$r_{\tau,8}^f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>entire period</td>
<td>-0.0545</td>
<td>-0.1475</td>
<td>-0.1164</td>
<td>-0.0755</td>
</tr>
<tr>
<td>crisis</td>
<td>-0.1460</td>
<td>-0.1669</td>
<td>-0.1484</td>
<td>-0.1127</td>
</tr>
</tbody>
</table>

Coefficient estimates $\hat{\beta}_\tau$ are not significant and negative for $\tau = 1, 2, 4, 8$, signalling the absence of statistically relevant relationship between index excess returns and future dividend strip excess returns (Table 2). Despite the size of the coefficient is not significant, its direction may deserve some attention: negative signs seem to underlie opposite tendencies between the two asset classes; the test results are indeed confirmed by the matrix of correlation in Table 3: index and future dividend strips are slightly negatively correlated for all the maturities involved.

It is quite interesting to observe that during the financial crisis (2008-2011) correlation coefficients are still negative, but higher in absolute values (despite the statistical significance does not improve); this different behaviour surely provides support to the hypothesis of considering dividends (and strategies on dividends) as a pure asset class, not exclusively dependent on index prices; furthermore it becomes feasible the possibility to consider derivative instruments on dividends not only for hedging purposes, but also as an alternative source of investment and risk diversification, especially in case of financial turmoil. In line with these findings it is also interesting to touch on an Eurex research that analyzes the relationship between dividends (as pure asset class) and other two traditional asset classes: bonds and equities. The dataset is composed by three samples of daily observations related to changes in EuroStoxx50 Index, 3-months Euribor and 10-year European government bond, from March 2006 to March 2008. Each sample is tested against the returns on December 2008 EuroStoxx50 Index Dividend Swap: results show very low correlations among factors, with squared-R statistics close to 1% for all the three regressions. It is not a case that markets for dividend-based derivatives are becoming more liquid: since 2008 trading volumes have significantly increased, allowing a process of standardization for some products, like equity-index dividend swaps, that started to be traded on regulated exchange platforms. In mid-2008, for example, dividend swaps on the EuroStoxx50 Index have been launched on Eurex: contracts traded in August 2011 reached the volume of 565,643 against a figure of 223,293 in August 2010: such a significant growth in volumes (+153%) means increasing demand and reduction in bid-ask spreads and in other pricing frictions; in few words: increasing market liquidity. During the financial crisis, despite the general prevailing of selling positions in financial markets, index dividend swaps have incurred in a growing demand by institutional investors (as buyers) that have become the ideal counterparties for banks, derivative dealers and hedge funds (sellers). In order to provide a pragmatic example of what is stated above, consider a portfolio manager who believes that in the near future expected S&P500 capital appreciation will overcome dividend income: he/she wants to “strip out” the dividend income stream of the equity investment in order to maximize the exposure to the index price appreciation; in doing

---

10when asset classes like equity and bond tend to co-move dissipating any benefits from diversification.
11mainly pension funds and insurance companies.
12all entities that need hedging against dividend risk, especially on the short-run.
13of course, also the vice versa holds.
so, he/she can synthetically sell dividends (maybe using a dividend swap) to buy more cash equities.13

Goldman Sachs[5] provides another way to see dividends and dividend-based products as a pure asset class; in a research paper underlines the fact that dividend policies are strictly related to two important features of a company: its *propensity to pay dividends* (focus on the will of distributing dividends, a company-specific decision) and its *ability to pay dividends* (related to the capacity of generating and increasing earnings). These two features may be left aside in a bond or equity valuation, or may have a less intense impact on price than other macro-economic factors.

In conclusion, dividend-based derivatives not only provide a way of hedging against dividend risk; they also allow dealers to separate their view on dividends from underlying price changes and then to isolate dividend returns from capital gains; furthermore such a view separation provides market players opportunities to invest in an asset class that is more closely related to specific company indicators than bond or equity ones: these are key sources of risk diversification.

### REFERENCES


Analytical Credit VaR under Different Scenarios for Probabilities of Default and Recoveries

In this article the authors present a unified framework for the analysis of the effects of concentration and contagion risk, overcoming two simplifying assumptions: static probabilities of default (PD) and recovery rates independent of the PDs. The model they introduce allows for non-static probabilities of default and recovery rates, while retaining the benefits of analytical computation.

Antonio CASTAGNA
Fabio MERCURIO
Paola MOSCONI

In recent years many models have been designed in theory and utilized in practice to calculate the value at risk (VaR) of credit portfolios. A very popular framework exploits an approximated analytical technique which applies to one-factor Merton type models. This method, originally introduced by Vasicek [10], consists in replacing the original portfolio loss distribution with an asymptotic one, whose VaR can be computed analytically. This approach, underlying the Internal Ratings-Based (IRB) framework of the Pillar I of Basel II [1] regulation, is based on the assumption of a homogeneous portfolio and a single systematic risk factor, thus neglecting concentration type risks.

The regulator is aware of the severe flaws of the model and forces financial institutions to equip themselves with more sophisticated tools to account for the omitted risks. Many steps have been taken in this direction, extending the original Vasicek result for homogeneous portfolios to include granularity risk, see e.g. Martin and Wilde [7], Gordy [6] and Tasche [4], sectoral concentration risk, see Pykhtin [9], and credit contagion, see e.g. Yun [12] and Bonollo et al [2].

In this paper, we push forward the analysis and study, in a unified framework, the effects of concentration and contagion risk, overcoming two simplifying assumptions that are common to the references above: i) static probabilities of default (PD); ii) recovery rates independent of the PDs. In fact, the model we present in what follows allows for non-static probabilities of default and recovery rates, while retaining the benefits of analytical computation.

The credit VaR framework we introduce, besides being a valid alternative to computationally heavy Monte Carlo simulations, contributes to the current debate on the overhaul of the Basel regulation, to avoid its pro-cyclical distortions caused by point-in-time PDs, by introducing through-the-cycle corrections, see also the Risk’s article [11]. In fact, on the one hand, our framework offers a sensible compromise for the controversial point regarding which level of PDs should be used in the VaR calculation, by allowing the consistent use of scenarios with specific PDs associated to different economic conditions. On the other hand, it accounts for all the risks considered both in the Pillar I and II of the Basel regulation, in a comprehensive fashion comparable to other general credit models.

General framework: introducing scenarios

The starting point is a multi-factor default-mode Merton model, in its extended multi-factor form
including the effects of contagion risk as proposed by Bonollo et al. [2]. Here we relax the assumption of constant probability of default and loss given default for each borrower, and allow them to assume values randomly drawn from a finite distribution, through the introduction of different possible scenarios.

Let us consider a loan’s portfolio, where the loans are associated to $M$ distinct borrowers. Each obligor has exactly one loan characterized by exposure $EAD_i$. We define the weight of a loan in the portfolio as $\bar{w}_i = EAD_i / \sum_{i=1}^{M} EAD_i$. The uncertainty on the creditworthiness of the $i$th borrower is modeled through the introduction of $S$ scenarios, each characterized by possible values which can be assumed by the default probability $PD_i$ and the loss-given-default $LGD_i = Q_i$ (where $Q$ stands for a stochastic variable with mean $\mu$ and standard deviation $\sigma$):

$$ (PD,Q)_i \sim \begin{cases} (p_{i1},Q_1) & \text{with probability } \lambda_{i1} \\ (p_{i2},Q_2) & \text{with probability } \lambda_{i2} \\ \vdots & \vdots \\ (p_{iS},Q_S) & \text{with probability } \lambda_{iS} \end{cases} \quad (1) $$

where $\sum_{s=1}^{S} \lambda_{is} = 1$ for each $i$. Each $Q$ is assumed to be independent of the other $Q$s and the remaining stochastic variables of the model.

Each obligor can thus be assigned different probabilities of default and losses given default in a corresponding number of scenarios. The total number of scenarios will be the result of all possible combinations of the obligors’ specific scenarios, which could in fact be rather large. In practice, though, obligors are gathered, on the basis of their creditworthiness, in a certain number of rating classes each featured by its own $PD$ and $LGD$, so that all the obligors belonging to the same class share the same $PD$ and $LGD$. By doing so, it is possible to drastically reduce the number of considered scenarios. The scenarios we will introduce are subject to either of the following two interpretations:

1. As a given state of the economy, with its specific probabilities of default and losses given default (or, equivalently, recovery rates) associated to each rating class. In each scenario, the obligors always belong to the same rating class, but rating class’ $PD$s and $LGD$s change with respect to other scenarios, due to cyclical conditions of the economy. As an example, a recession can give rise to a scenario with generally higher $PD$s and $LGD$s than a growth period.

2. As a given state of economy, with rating classes containing a given set of obligors. In this case, for each rating class, $PD$s and $LGD$s are constant through all the possible scenarios, but in each scenario the composition (in terms of obligors) of each single rating class is different. In practice, it is as if we were modeling the rating migration of the obligors.

If scenarios are built according to the first perspective, in each of them the number of $PD$s and $LGD$s is reduced from $M$ to the number of rating classes (usually below 20). When instead the second perspective is used, we cannot abate substantially the number of scenarios unless we make strong assumptions on the possible migrations of the single obligors. Example of such assumptions are shown in Section 2.

Finally, the above framework allows for an (implicit) correlation between the level of the default probabilities and the losses given default, simply by devising scenarios where, for instance, higher $LGD$s are associated to higher $PD$s.

Model description

We here sketch the theoretical model, referring to the extended version of this work (Castagna et al. [3]) for details and discussion. First, we present the multi-factor framework, then we add the contagion part.

We assume that asset returns $\{X_i\}_{i=1,..,M}$ are the key variables to be modeled: default occurs for borrower $i$, in a given scenario $\varphi$, when the corresponding $X_{i\varphi}$ falls below the threshold $N^{-1}(p_{i\varphi})$. Asset returns are assumed to be distributed according to a standard normal distribution:

$$ X_{i\varphi} = r_{i\varphi} Y_i + \sqrt{1 - r_{i\varphi}^2} \xi_i \quad (2) $$

where the systematic contribution is expressed in terms of a composite variable $\{Y_i\}_{i=1,..,M}$, encoding the effects of multiple sectors and the idiosyncratic component of risk is given by $\xi_i \sim N(0,1)$ independent of $Y_i$. The sensitivity of borrower $i$ to systematic risk, namely $r_{i\varphi} \geq 0$, depends in general on the given scenario.

As is well known, the composite factor $Y_i$ can be expressed as a linear combination of $N$ independent systematic factors $Z_k \sim N(0,1)$ with $k = 1,\ldots,N$, and the assumption of unit variance yielding $\sum_{k=1}^{N} a_{ik}^2 = 1$: \( Y_i = \sum_{k=1}^{N} a_{ik} Z_k \) \quad (3)

In turn, the quantity $r_{i\varphi}Y_i$ can be rewritten in terms of a unique systematic risk factor $\bar{Y} = \sum_{k=1}^{N} b_k Z_k$, with $b_k \geq 0$, and a residual contribution
The conditional correlation between distinct obligors $i$ and $j$, respectively in scenarios $\varphi$ and $\psi$, assumes the form of:

$$
\rho_{Y_{\varphi}, Y_{\psi}} = \frac{r_{iq} r_{jq} \sum_{k=1}^{N} \gamma_{ik} \gamma_{jk} - a_{iq} a_{jk}}{\sqrt{(1 - a_{iq}^2)(1 - a_{jq}^2)}}
$$

The unconditional correlation between borrowers $i$ and $j$, respectively in scenarios $\varphi$ and $\psi$, is given by $\text{corr}(X_{i\varphi}, X_{j\psi}) =: \rho_{i\varphi, j\psi} = r_{iq} r_{jq} \sum_{k=1}^{N} \beta_{ik} \beta_{jk}$.

$$
\sum_{k=1}^{N} (r_{iq} a_{ik} - a_{iq} b_{k}) Z_{k} \text{ independent of } Y_{\varphi} \text{ which leads to the conditional asset correlation:}^{14}
$$

$$
\rho_{Y_{\varphi}, Y_{\psi}} = \frac{r_{iq} r_{jq} \sum_{k=1}^{N} \alpha_{ik} \alpha_{jk} - a_{iq} a_{jk}}{\sqrt{(1 - a_{iq}^2)(1 - a_{jq}^2)}}
$$

The non-negative coefficients $a_{iq} := r_{iq} \sum_{k=1}^{N} a_{ik} b_{k}$ are effective factor loadings, obtainable through an optimization procedure. The unit variance constraint imposes that $\sum_{k=1}^{N} b_{k}^2 = 1$.

As for the contagion risk, we assume that obligors are broadly divided into two categories: those firms which are immune from contagion (referred to as “I-firms”, i.e. infecting) and those companies which can be contaminated by the first group through credit contagion (“C-firms”). Asset returns associated to group “I” follow the multi-factor specification given by eq. (2) while “C-firms” asset returns are assumed to satisfy (for obligor $i$ in a scenario $\varphi$):

$$
X_{i\varphi} = r_{iq} Y_{i\varphi} + \sqrt{1 - r_{iq}^2} \xi(\Gamma_{i\varphi}, \epsilon_{i\varphi})
$$

The firm-specific factor $\xi(\Gamma_{i\varphi}, \epsilon_{i\varphi})$, which is assumed to be scenario-independent, is defined by:

$$
\xi(\Gamma_{i\varphi}, \epsilon_{i\varphi}) = g_i \Gamma_{i\varphi} + \sqrt{1 - C_i^2} \epsilon_{i\varphi}
$$

with the unit variance property of $X_{i\varphi}$ being preserved by assuming $\sum_{k=1}^{N} \gamma_{ik}^2 = 1$. We decompose each sector into a “I” segment and a “C” one. Therefore, the contagion effect experienced by an arbitrary “C-firm” can be thought of as the weighted sum of contributions from the infecting segments of different sectors. Under this specification, the number of latent contagion factors equals the number of industry-geographic factors, $N$. The coefficient $g_i$ plays the role of a contagion factor loading and represents a measure of how much obligor $i$ is overall affected by contagion.

In this framework, asset returns follow:

$$
X_{i\varphi} = a_{iq} Y_{i\varphi} + \sum_{k=1}^{N} (r_{iq} a_{ik} - a_{iq} b_{k}) Z_{k} + \sqrt{1 - r_{iq}^2} \sum_{k=1}^{N} \gamma_{ik} C_{k} + \sqrt{1 - r_{iq}^2} \sqrt{1 - S_i^2} \epsilon_{i\varphi}
$$

VaR decomposition and results

Given this setup, the portfolio loss rate $L$ can be written as the weighted sum over individual loss rates

$$
L = \sum_{i=1}^{M} w_{i} L_{i}
$$

Each $L_{i}$ is a stochastic variable whose value varies across different scenarios:

$$
L_i = \begin{cases} 
Q_{\lambda_1} 1_{(X_{i\varphi} \leq N^{-1}(p_{\varphi}))} & \text{with probability } \lambda_1 \\
Q_{\lambda_2} 1_{(X_{i\varphi} \leq N^{-1}(p_{\psi}))} & \text{with probability } \lambda_2 \\
\vdots & \vdots \\
Q_{\lambda_M} 1_{(X_{i\varphi} \leq N^{-1}(p_{M}))} & \text{with probability } \lambda_M 
\end{cases}
$$

$Q_{\lambda_{1}}$ and the indicator function $1_{(\cdot)}$ represent respectively the stochastic LGD and the event of default, associated to obligor $i$, in a given scenario.

Our goal is to calculate explicitly the quantile $t_q(L)$, at confidence level $q$ of the quantity $L$. Following the idea in Pykhtin [9]), we calculate $t_q(L)$ through a Taylor expansion around the quantile of another variable $\overline{L}$, such that $t_q(\overline{L})$ is analytical and sufficiently close to $t_q(L)$.

We define the variable $\overline{L}$ as the limiting loss distribution in the one-factor Merton framework [8], i.e.:

$$
\overline{L} \equiv l(\overline{Y}) = E[L|\overline{Y}]
$$

Calculating the expectation explicitly, we get:

$$
\overline{Y} = E \left[ \sum_{i=1}^{M} w_{i} Q_{i} 1_{(X_{i\varphi} \leq N^{-1}(p_{i\varphi}))} \right] = \sum_{i=1}^{M} w_{i} \sum_{\varphi=1}^{N} \lambda_{iq} h_{iq} P(X_{i\varphi} \leq N^{-1}(p_{i\varphi})|\overline{Y})
$$

$$
= \sum_{i=1}^{M} w_{i} \sum_{\varphi=1}^{N} \lambda_{iq} h_{iq} N \left[ \frac{N^{-1}(p_{i\varphi}) - a_{iq} \overline{Y}}{\sqrt{1 - a_{iq}^2}} \right]
$$

$$
= \sum_{i=1}^{M} w_{i} \sum_{\varphi=1}^{N} \lambda_{iq} h_{iq} P_{i\varphi}(\overline{Y})
$$

Winter 2014

14The unconditional correlation between borrowers $i$ and $j$, respectively in scenarios $\varphi$ and $\psi$, is given by $\text{corr}(X_{i\varphi}, X_{j\psi}) =: \rho_{i\varphi, j\psi} = r_{iq} r_{jq} \sum_{k=1}^{N} \alpha_{ik} \alpha_{jk}$.
where \( \hat{p}_{iq}(y) \) is the probability of default of borrower \( i \), given scenario \( \varphi \), conditional on \( Y = y \):

\[
\hat{p}_{iq}(y) = N \left[ \frac{N^{-1}(p_{iq}) - a_{iq}y}{\sqrt{1 - a_{iq}^2}} \right]
\]

with \( N \) denoting the cumulative normal distribution and \( N^{-1} \) its inverse. The quantile of \( L \) at level \( q \) can be calculated analytically as:

\[
t_q(L) = l(N^{-1}(1 - q))
\]

It can be shown, see Castagna et al. [3], that the first order derivative vanishes automatically, so that its systematic and idiosyncratic components:

\[
\nu(y) = v_\varphi(y) + v_{GA}(y)
\]

where:

\[
\Delta t_q = -\frac{1}{\sqrt{\pi y}} \left[ v'(y) - v(y) \left( \frac{v'(y)}{v(y)} + y \right) \right] \bigg|_{y = N^{-1}(1 - q)}
\]

The function \( l(y) \) is defined as in (9), (10), and \( v(y) = \varphi [L|Y = y] \) is the variance of \( L \) conditional on \( Y = y \), which can be decomposed in terms of its systematic and idiosyncratic components:

\[
v(y) = v_\varphi(y) + v_{\varphi,LA}(y)
\]

where:

\[
v_\varphi(y) = \varphi[E(L|\{Z_k\})|Y = y]
\]

\[
v_{\varphi,LA}(y) = E[\varphi(L|\{Z_k\})|Y = y]
\]

Formulæ (11) through (14) encodes the effects of concentration risk.

Explicit expressions for the corrections (14) can be derived in a relatively easy fashion:

\[
v_\varphi(y) = \sum_{i,j=1}^M \sum_{\varphi = 1}^S \sum_{\psi = 1}^S \mu_{iq}\mu_{ij} \hat{\rho}_{iq}(y)
\]

\[
* \lambda_{iq,ij} N_2(N^{-1}(\hat{p}_{ij}(y)), N^{-1}(\hat{p}_{ij}(y)), \rho_{ij,\varphi,LA}) +
\]

\[
- \sum_{i,j=1}^M \sum_{\varphi = 1}^S \sum_{\psi = 1}^S \mu_{ij}\mu_{ij} \lambda_{iq,ij} \hat{\rho}_{ij}(y) \hat{\rho}_{ij}(y)
\]

where \( N_2 \) denotes the bivariate normal cumulative distribution function and \( \lambda_{iq,ij} \) is the joint probability that obligor \( i \) assumes values in scenario \( \varphi \) and obligor \( j \) in scenario \( \psi \), and

\[
v_{\varphi,LA}(y) = \sum_{i,j=1}^M \sum_{\varphi = 1}^S \sum_{\psi = 1}^S \lambda_{iq,ij} N_2(N^{-1}(\hat{p}_{ij}(y)), N^{-1}(\hat{p}_{ij}(y)), \rho_{ij,\varphi,LA}) +
\]

\[
- \sum_{i,j=1}^M \sum_{\varphi = 1}^S \sum_{\psi = 1}^S \mu_{ij}\mu_{ij} \lambda_{iq,ij} \hat{\rho}_{ij}(y) \hat{\rho}_{ij}(y)
\]

\[
\rho_{ij,\varphi,LA} \]

The conditional correlation \( \rho_{ij,\varphi,LA} \) appearing in eq.s (15), (16) is given by formula (7).

The final result can be stated as follows: the quantile \( t_q(L) \) at level \( q \) of the loss distribution \( L \) is given by the approximated formula (11), where the asymptotic zeroth order term \( t_0(L) \) is expressed by eq.s (10), (10), and the correction \( \Delta t_q \) is encoded into equations (12), (13), (15) and (16).

**Numerical analysis**

This section is devoted to the numerical implementation of the theoretical model. The credit portfolio and the model are specified as follows\(^1\):

- Given loan exposures assigned following the empirical rule \( EAD_i = \hat{\rho} \) (see [3] and references therein), we assume that the last (in terms of notional) 20% of obligors belongs to the group of infecting “I-firms”.

- We consider \( N = 11 \) industry-geographic sectors. We assume them to be standardized but dependent on each other through an appropriate correlation matrix derived from MSCI EMU industry indices. Each obligor is associated only with one sector.

- For the contagion specification we use the same structure of sectors. For each obligor belonging to class “C”, we consider both the impact of the infecting segment of each sector onto it and its overall sensitivity to contagion (encoded into the discretionary parameter \( g_i \)). We assume that only two contagion sectors affect each obligor.

- Obligors are grouped into rating classes. Table 1 summarizes the properties of the starting configuration of such aggregation, as proposed by Gordy [5]. We have chosen the \( LGD \) mean values so as to have on average \( \rho^{0.30} = 30\% \). The corresponding standard deviations are set to \( \sigma^0 = 1/2 \sqrt{\rho^0(1-\rho^0)} \). For

\(^1\)Explicit expressions for the derivatives of \( l(y) \) and \( v(y) \) can be found in Appendix C2 of Castagna et al. [3].

\(^2\)More details are in Castagna et al. [3].
TABLE 1: Rating classes according to Gordy [5].

<table>
<thead>
<tr>
<th>Scenario</th>
<th>M</th>
<th>$t_{99.9%}(L)$</th>
<th>$t_{99.9%}(\bar{L})$</th>
<th>$\Delta_{YC}$</th>
<th>$\Delta_{G\bar{A}}$</th>
<th>$\Delta_{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sgl</td>
<td>0.0553</td>
<td>0.0357</td>
<td>0.0196</td>
<td>0.0071</td>
<td>0.0120</td>
<td>0.0076</td>
</tr>
<tr>
<td>1A</td>
<td>0.0635</td>
<td>0.0378</td>
<td>0.0256</td>
<td>0.0076</td>
<td>0.0125</td>
<td>0.0132</td>
</tr>
<tr>
<td>1B</td>
<td>0.0618</td>
<td>0.0385</td>
<td>0.0233</td>
<td>0.0076</td>
<td>0.0125</td>
<td>0.0108</td>
</tr>
<tr>
<td>1C</td>
<td>0.1086</td>
<td>0.0468</td>
<td>0.0618</td>
<td>0.0093</td>
<td>0.0141</td>
<td>0.0477</td>
</tr>
<tr>
<td>Sgl</td>
<td>0.0526</td>
<td>0.0413</td>
<td>0.0113</td>
<td>0.0066</td>
<td>0.0042</td>
<td>0.0070</td>
</tr>
<tr>
<td>1A</td>
<td>0.0603</td>
<td>0.0436</td>
<td>0.0167</td>
<td>0.0070</td>
<td>0.0044</td>
<td>0.0123</td>
</tr>
<tr>
<td>1B</td>
<td>0.0588</td>
<td>0.0444</td>
<td>0.0144</td>
<td>0.0070</td>
<td>0.0044</td>
<td>0.0101</td>
</tr>
<tr>
<td>1C</td>
<td>0.1023</td>
<td>0.0535</td>
<td>0.0488</td>
<td>0.0085</td>
<td>0.0050</td>
<td>0.0438</td>
</tr>
</tbody>
</table>

TABLE 2: Results obtained for an average quality portfolio, characterized by 7 rating classes, 11 industry-geographic areas and contagion factors, at the level of confidence $q = 99.9\%$.

notational purposes we group the $PD$ and $LGD$ values into vectors of 7 elements:
$$p^0 = (p^0_{AAA}, p^0_{AA}, \ldots, p^0_{CCC})$$
$$\mu^0 = (\mu^0_{AAA}, \mu^0_{AA}, \ldots, \mu^0_{CCC})$$
$$\sigma^0 = (\sigma^0_{AAA}, \sigma^0_{AA}, \ldots, \sigma^0_{CCC})$$

In the scenarios’ definition, we assume that obligors are perfectly correlated with each other. Therefore, the behavior of a single obligor $i$ in terms of his evolution towards a scenario $\varphi$ describes as well the behavior of all the other obligors.

The model, in theory, deals with scenarios defined for each single obligor. However, following the common practice, we simplify the problem by aggregating obligors into rating classes, thus effectively analyzing joint scenarios, referring to whole classes of rating: this significantly lightens the computational burden, without being an unrealistic choice. In addition, we limit ourselves to three distinct scenarios.

We propose two different implementations and besides, for comparison’s purposes, we introduce also the special case when only one scenario is present, characterized by the vectors $(p^0, \mu^0, \sigma^0)$ and probability weight $\lambda_1 = 1$. The distribution of obligors onto the different rating classes (i.e. the elements of $(p^0, \mu^0, \sigma^0)$) is assumed to be that of an average quality portfolio, such that speculative grade loans account for 50% of the total exposure. The single scenario case, which we label by “Sgl”, serves as a reference to measure the contribution of our assumed uncertainty in rating features (as expressed by the different scenarios).

Fig. 2 shows the decomposition of the approximated VaR in terms of its main contributions, i.e. the zeroth order term $t_{q}(\bar{L})$ and the corrections due to granularity in the exposures and the multi-factor setup. While the asymptotic VaR is an increasing function of the number of obligors, the resulting downward sloping curves in Fig. 1 are due to the effects of second order corrections. As the number of obligors increases, $\Delta_{\infty}$ tends towards a steady value, while the granularity adjustment becomes progressively negligible (see Table 2). When analyzing the influence of different scenario choices, we notice that the major role is played by the zeroth order term $t_{q}(\bar{L})$ and the second order correction $\Delta_{\infty}$, the granularity adjustment being only mildly affected by it.

We have performed an analogous analysis on the second type of scenarios, which involve transitions across rating classes. The results are displayed in Fig. 3.

As for the other two implementations, we adhere to the two ways of looking at the scenarios’ building already introduced in Section 2. In more
**FIGURE 1:** Approximated VaR, \( t_q(L) \), at level of confidence \( q = 99.9\% \) vs number of obligors \( M \), for different scenarios of type 1.

**FIGURE 2:** Decomposition of the second order VaR, \( t_q(L) \), into its main components: zeroth order term \( t_q(L) \equiv l(y) \), with \( y = N^{-1}(1-q) \) and \( q = 99.9\% \), granularity adjustment \( \Delta_{GA} \) and multi-factor correction \( \Delta_{\infty} \).

**FIGURE 3:** Approximated VaR, \( t_q(L) \), at level of confidence \( q = 99.9\% \) vs number of obligors \( M \), for different scenarios of type 2.
2. We interpret scenarios in terms of a particularly simple kind of migration. As initial state, at time $t = 0$, we choose the single scenario setup. Changes occur at time $t = 0^+$ and can be seen as joint migrations of obligors towards other rating classes. This specification is a simplified case of the general setting including all the possible scenarios ($\binom{7+1}{2}$ in our case) related to the obligors’ migrations. To this end, it proves useful to introduce operators which define the transition from one rating class to another. Let us define $\hat{P}_k$, $k = \pm 1 \ldots \pm 7$ such that:

$$
\hat{P}_{k+1}^{j} = P_{j}^{0} \text{ and } \hat{P}_{k-1}^{j} = P_{j}^{0}
$$

where $j$ indicates the rating class ($j = 1$ corresponds to AAA, $j = 2$ to AA and so on) and we adopt the convention on the signs such that $-k$ stands for an improvement of $k$ rating classes and $+k$ for a deterioration of $k$ classes. Border values on the rating scale deserve a special treatment. For example, we set:

$$
\hat{P}_{-1}^{0} = P_{1}^{0} \text{ and } \hat{P}_{1}^{0} = P_{1}^{0}
$$

and so on. We then consider the following cases. At $t = 0^+$:

We start from scenarios of type 1. Table 2 summarizes the main results, including both the final calculation of the approximated VaR and of its constituent components.

From the data collected in Table 2, we can visualize in Fig. 1 the behavior of the approximated VaR, $t_q(L)$ ($q = 99.9\%$), versus the number of obligors $M$, for different scenarios. The red line represents the single scenario situation and, given the cases analyzed, it is associated to the lowest VaR. All other cases, except from the most conservative 1C, deviate from the (Sgl) single scenario’s VaR, of about $10 \div 15\%$. Case 1C, which entails doubling and tripling of the PDs values, shows a variation of VaR which is roughly twice as much as the value in the single scenario case.

As expected, though border classes are only partially affected by the presence of different scenarios, on average the values of the PDs and LGDs vary more drastically in this framework, leading to higher values of the approximated VaR. In the cases considered, the single scenario value at risk appears to be doubled or even tripled in the most conservative case 2C.

### Scenario Analysis

**VaR $t_q(L)$ vs mixture of single-scenario VaRs**

Given our scenarios formulation, one may be tempted to calculate the resulting VaR simply as a weighted sum of the VaRs in the individual scenarios. Our previous calculations show that the formula for $t_q(L)$ is indeed different and not trivially obtained by mixing single-scenario VaRs. We now assess the discrepancy between these two evaluations, comparing the results obtained in our framework, through eqs (11) and (12), with the sum of single-scenario VaRs, namely $[t_q(L)]^\varphi$, each weighted by the appropriate probability $\lambda_\varphi$:

$$
\text{weighted sum} = \sum_{\varphi=1}^{S} \lambda_\varphi [t_q(L)]^\varphi.
$$

Table 3 shows the outcomes for different scenarios of type 1 and 2. The last column reports the values of the percentage difference obtained as

$$
\Delta\% = \frac{t_{99.9\%}(L) - \text{weighted sum}}{t_{99.9\%}(L)} \cdot 100\%.
$$

Such a quantity is of the order of $5 \div 10\%$ for scenarios which are not too distant from the single
FIGURE 4: Approximated VaR, \( t_{q}(L) \) with \( q = 99.9\% \), for different scenarios of type 1, corresponding to a portfolio of loans homogeneously distributed across sectors and a portfolio concentrated in two sectors (\( M = 500 \)).

TABLE 3: Comparison between the approximated VaR, \( t_{99.9\%}(L) \), and the weighted sum of individual VaRs corresponding to different scenarios (of type 1 and 2) for \( M = 500 \).

<table>
<thead>
<tr>
<th>Scenario</th>
<th>( t_{99.9%}(L) )</th>
<th>weighted sum</th>
<th>( \Delta % )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>0.0603</td>
<td>0.0553</td>
<td>8.33%</td>
</tr>
<tr>
<td>1B</td>
<td>0.0588</td>
<td>0.0562</td>
<td>4.52%</td>
</tr>
<tr>
<td>1C</td>
<td>0.1023</td>
<td>0.0664</td>
<td>35.10%</td>
</tr>
<tr>
<td>2A</td>
<td>0.0754</td>
<td>0.0553</td>
<td>26.65%</td>
</tr>
<tr>
<td>2B</td>
<td>0.1099</td>
<td>0.0598</td>
<td>45.58%</td>
</tr>
<tr>
<td>2C</td>
<td>0.1624</td>
<td>0.0728</td>
<td>55.17%</td>
</tr>
</tbody>
</table>

TABLE 4: Approximated VaR, \( t_{99.9\%}(L) \), and its components, for different choices of the number of contagion sectors (2,5 or 11), number of obligors \( M \) and scenarios Sgl (single-scenario) and 1B.

<table>
<thead>
<tr>
<th>Ctg. sectors</th>
<th>Scenario</th>
<th>M</th>
<th>( t_{99.9%}(L) )</th>
<th>( t_{99.9%}(\bar{L}) )</th>
<th>( \Delta_{YC} )</th>
<th>( \Delta_{C} )</th>
<th>( \Delta_{C/A} )</th>
<th>( \Delta_{s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Sgl</td>
<td>500</td>
<td>0.0526</td>
<td>0.0413</td>
<td>0.0113</td>
<td>0.0066</td>
<td>0.0042</td>
<td>0.0070</td>
</tr>
<tr>
<td>2</td>
<td>Sgl</td>
<td>1000</td>
<td>0.0580</td>
<td>0.0413</td>
<td>0.0168</td>
<td>0.0121</td>
<td>0.0042</td>
<td>0.0125</td>
</tr>
<tr>
<td>2</td>
<td>1B</td>
<td>500</td>
<td>0.0516</td>
<td>0.0428</td>
<td>0.0088</td>
<td>0.0071</td>
<td>0.0024</td>
<td>0.0029</td>
</tr>
<tr>
<td>2</td>
<td>1B</td>
<td>1000</td>
<td>0.0569</td>
<td>0.0428</td>
<td>0.0140</td>
<td>0.0115</td>
<td>0.0021</td>
<td>0.0067</td>
</tr>
<tr>
<td>2</td>
<td>1B</td>
<td>1000</td>
<td>0.0578</td>
<td>0.0428</td>
<td>0.0049</td>
<td>0.0070</td>
<td>0.0021</td>
<td>0.0028</td>
</tr>
<tr>
<td>2</td>
<td>1B</td>
<td>1000</td>
<td>0.0588</td>
<td>0.0444</td>
<td>0.0144</td>
<td>0.0024</td>
<td>0.0021</td>
<td>0.0029</td>
</tr>
<tr>
<td>2</td>
<td>1B</td>
<td>1000</td>
<td>0.0544</td>
<td>0.0444</td>
<td>0.0100</td>
<td>0.0026</td>
<td>0.0044</td>
<td>0.0056</td>
</tr>
<tr>
<td>2</td>
<td>1B</td>
<td>1000</td>
<td>0.0537</td>
<td>0.0460</td>
<td>0.0118</td>
<td>0.0067</td>
<td>0.0021</td>
<td>0.0097</td>
</tr>
</tbody>
</table>
scenario case (e.g. 1A and 1B), but becomes extremely relevant for conservative ones, the discrepancy being more pronounced for scenarios of type 2, where $\Delta _C$ ranges from about 26% to 55%. Therefore, the trivial approach based on calculating the weighted sum over different scenarios may significantly underestimate the true (and also the second order approximated) value at risk.

### Sector Concentration Analysis

The study of the previous paragraph has been conducted assuming a portfolio of loans homogeneously distributed across industry-geographic areas. Departing from the portfolio of loans homogeneously distributed, we now consider a portfolio concentrated mainly in two sectors. The comparison between the approximated VaR obtained in this case and the one corresponding to a uniformly distributed portfolio is shown in Fig. 4 for $M = 500$ obligors, and scenarios of the first type.

The interpolated VaR points referring to scenarios, ranging from Sgl to 1C, produce two almost parallel curves. Therefore, the effect of sector concentration produces a roughly constant shift in $t_q(L)$, resulting in higher values for more concentrated portfolios, as expected.

### Contagion Analysis

We conclude the numerical analysis by studying the effects of contagion. In particular, we focus on the role played by the number of infecting segments acting on each obligor. In the previous analysis, to simplify things, we opted for just two contagion sectors. Here we compare the previous results with those obtained by incrementing the number of contagion sectors. Explicitly, we consider five and eleven sectors, in addition to the original two, assuming for simplicity each obligor is uniformly impacted by such infecting segments. To keep things general we choose two scenarios, Sgl and 1B, and number of obligors $M = 500$ and 1000. The complete results are collected in Table 4 while Fig. 5 highlights the main aspects.

The picture shows the behavior of $t_q(L)$ ($q = 99.9\%$) and of the total contagion correction $\Delta _C$ versus the number of contagion sectors (we connected the three points with lines). A peak occurs in correspondence of the intermediate number of sectors (in this case five). This is consistent with intuition. Starting from a low number of sectors, when increasing it, the effects of contagion become more relevant, till the moment in which sector diversification starts to predominate, thus leading to a reduction of the contagion adjustment and consequently of the total VaR. This effect is particularly evident here, given our choice involving uniformly distributed participation weights.

### Conclusions

Besides delving into the well known issues regarding concentration risk (single-name, sector concentration and contagion), we propose a new perspective which allows to model in a more flexible and non static way the creditworthiness (PDs and LGDs) of obligors, and the link between the level of default probabilities and the losses given default. This is achieved through the introduction of different pos-
sible scenarios, each characterized by distinctive rating features and weight.

Each obligor, initially assigned to a rating class, at a successive instant of time can change his rating properties according to a given set of scenarios. As a byproduct, PDs and LGDs which are assumed independent in each single scenario, turn out to be implicitly correlated in the wider picture.

If properly chosen, scenarios are also apt to implicitly define, for each obligor, the probability of transition to a rating class different than the initial one. At the portfolio level, obligors can then be grouped according to their features into distinct rating classes. Our model, therefore, can also be interpreted as a rating migration one, being suitable to describe the joint evolution of such classes towards others (implicitly defining the corresponding joint transition probabilities).

The examples we have considered in the paper indicate a noteworthy increment of the VaR of a credit portfolio in the presence of uncertainty in the rating features.

ABOUT THE AUTHORS
Antonio Castagna is Senior Consultant at Iason.
Email address: antonio.castagna@iasonltd.com
Fabio Mercurio is working at Bloomberg LP.
Paola Mosconi is working at Banca IMI; at the time of the paper writing she was a consultant at Iason.

ABOUT THE ARTICLE
Submitted: January 2013.
Accepted: March 2013.

References


34
Optimal Quantization Method

Application to Multidimensional and Path-dependent Problems in Finance

In this article, we propose Optimal Quantization Method (hereafter OqM) to numerically solve multidimensional and multistep problems in finance. A special emphasis is made on the computational aspects and the numerical applications: the aim is to demonstrate how you can use OqM to reduce computational costs of classical Monte Carlo simulation approach.

Gaetano MARINO

Everything that can be counted does not necessarily count; everything that counts cannot necessarily be counted.
Albert Einstein

Generally quantization of random vectors can be considered as a discretization of the probability space by at most N values providing in some sense the best approximation to the original distribution. Let $X$ be a random vector on a probability space $(\Omega, \mathcal{A}, P)$ taking its value in $\mathbb{R}^d$, we denote by $P_X$ its distribution on $\mathbb{R}^d$. Quantization consist in approximating $X$ by a random variable $Y$ taking finitely many value in $\mathbb{R}^d$. The finite set $\Gamma = \{y_1, \ldots, y_N\}$ is often called a $N-$quantizer of $X$ (or $N-$codebook). In numerical application $\Gamma$ is also called grid. Then it become essential to optimize the geometric location of these quantizers for a given distribution and to evaluate the resulting error. Some numerical procedures have been developed in order to get optimal quadratic quantization of the Gaussian distribution in high dimension: in particular, the Newton's method in the deterministic case (dimension $d = 1$) and stochastic gradient method in higher dimensional case ($d \geq 2$). In a more mathematical form, the problem is to find out a measurable function $Y$ such that:

$$
\|X - Y\|_p = \inf \left\{ \|X - Y\|_p, \ Y : \mathbb{R}^d \to \mathbb{R}^d, \ card \ (Y (\Omega)) \leq N, \ N \in \mathbb{N}^* \{1, 2, \ldots \} \right\}
$$

This problem has been initially investigated for its applications to signal transmission. The point of interest was to design the random variable $Y$ in order to minimize the resulting error for a fixed quantization level $N$: this led to the concept of optimal quantization. More recently, quantization was introduced in numerical probability to devise numerical integration methods and to solve multidimensional stochastic control problems. The infinite dimensional setting has been extensively investigated from both theoretical and numerical viewpoints with a special attention paid to functional quantization. Bi-misurable stochastic processes are viewed, e.g., as random variable taking values in their path space such as $L^2_T := L^2 ([0, T], dt)$. In this paper, we focus on the purely quadratic framework, essentially because it is a natural (and somewhat easier) framework for the computation of optimized grids for the Brownian motion and for the first applications, like the pricing of path-dependent option. It will be very interesting to notice that for a centered Gaussian on a Hilbert space, it suffices to know only the eigenvalues $(\lambda_n)_{n \geq 1}$ of the covariance operator, so that the quantization problem for the Gaussian can be completely solved by the investigation of an auxiliary quantization problem for $\bigotimes_{n=1}^{\infty} \mathcal{N} (0, \lambda_n)$.

Quantization of a random variable

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $E$ a separable reflexive Banach space; the norm of $E$ is denoted $|\cdot|$. The basic idea of OqM of a random variable $X$
When the minimum is reached, one refers to optimal quantization. Let us consider the settled point set \( \Gamma = \{y_1, \cdots, y_N\} \subset E \) be a settled point set and \( C = \{C_1, \cdots, C_N\} \) the associated Voronoi partition. A nearest neighbor projection on \( \Gamma \) is a Borel application \( \text{Proj}_\Gamma : E \to \Gamma \) such that:

\[
\forall x \in E, \quad |x - \text{Proj}_\Gamma x| = \min_{i \in \{1, \cdots, N\}} |x - y_i|.
\]

Let \( X \) be a \( L^p \) random variable taking its values in \( E \) and \( Y \) taking its values in the settled point set \( \Gamma = \{y_1, \cdots, y_N\} \subset E, N \in \mathbb{N}^+ \). If the random variable \( X^\Gamma := \text{Proj}_\Gamma (X) \) is a nearest neighbor projection on \( \Gamma \), called Voronoi \( \Gamma \)–quantizer of \( X \), we clearly have:

\[
\|X - X^\Gamma\|_p \leq \|X - Y\|_p
\]

Then, solving the minimization problem \( (1) \) comes to solving the simpler minimization problem:

\[
\min \left\{ \|X - \text{Proj}_\Gamma\|_p \right\}, \quad \Gamma \in E, \quad \text{card} (\Gamma) \leq N, \in \mathbb{N}^+
\]

where \( \|X - \text{Proj}_\Gamma\|_p \) is the \( L^p \)–quantization error.

When the minimum is reached, one refers to optimal quantization. So, if one identifies a grid \( \Gamma \) of size \( N \) with the \( N \)–tuple \( \{y_1, \cdots, y_N\}, \ y_i \in \mathbb{R}^d \), the \( p^{th} \) power of the \( L^p \)–quantization error is a symmetric function defined on \( N \)–tuple with pairwise distinct components by:

\[
\mathcal{E}_N^p (X) := \int_{\mathbb{R}^d} e_N^p (y, \xi) \mathbb{P}_X (d\xi)
\]

where \( \mathbb{P}_X \) is the distribution of \( X \) and \( e_N^p (y, \xi) := \min_{i \in \{1, \cdots, N\}} |y_i - \xi|^p \), \( y_i, \xi \in \mathbb{R}^d \). The function \( |\cdot|^p \) is sometimes called local quantization error.

The quadratic quantization error, to the power 2, that can be achieved when approximating \( X \) by a quantizer of level \( N \) is often called (quadratic) distortion and is denoted \( \mathcal{E}_N (X) \). Then, a question naturally arise: how does the minimum behave as \( N \) goes to infinity? When \( \text{card} (X (\Omega)) \) is infinite, this minimum strictly decreases to 0 as \( N \) goes to infinity. The rate of convergence is ruled by Zador theorem in the finite dimensional case and completed by Bucklew & Wise and Graf & Luschgy:

**Theorem:** Assume \( X \in L^{p+\epsilon} (\mathbb{P}) \) for some \( \epsilon > 0 \). Let \( \mathbb{P}_X (d\xi) = \phi (\xi) \lambda_d d\xi + \mu (d\xi) \) be the canonical decomposition of the distribution of \( X \), where \( \phi := \frac{d\mathbb{P}_X}{d\lambda_d} \) is the Radon-Nikodym density of \( \mathbb{P}_X \) with respect to the Lebesgue measure \( \lambda_d \) on \( \mathbb{R}^d \), \( \lambda_d \perp \mu \). Then, (if \( \phi \neq 0 \)), the \( L^p \)–quantization error of level \( N \) satisfies:

\[
\lim_{N \to +\infty} \left( N^\frac{p}{2} \min_{\text{card} \( (\Gamma) \)} \|X - X^\Gamma\|_p \right) = \int_{\mathbb{R}^d} \phi \frac{d\mathbb{P}_X}{d\lambda_d} (d\xi) \lambda_d (d\xi)
\]

\[
\mathcal{E}_{N,p} (X, \mathbb{R}^d) \sim_{N \to \infty} \int_{\mathbb{R}^d} \phi \frac{d\mathbb{P}_X}{d\lambda_d} (d\xi) \lambda_d (d\xi)
\]

where:

\[
\|\phi\|_{\frac{d\mathbb{P}_X}{d\lambda_d}} \equiv \left( \int_{\mathbb{R}^d} \phi \frac{d\mathbb{P}_X}{d\lambda_d} (d\xi) \lambda_d (d\xi) \right)^\frac{\frac{d\mathbb{P}_X}{d\lambda_d}}{p}
\]
The covariance operator

The aim of this section is to describe a numerical component analysis. Let variable: quantizer is stationary. Then, if \( Y \) \( \sim \) \( \mathcal{N} \) of the quantization problem of \( X \) and the quantization error of the projected random variable: is stationary (or self-consistent) if \( Y = \mathbb{E} [X|Y] \). A quadratic optimal quantizer is stationary. Then, if \( Y = \text{Proj}_J(X) \) is an \( L^2 \) optimal quantizer and \( C = \{ C_1, \ldots, C_n \} \) is the associated Voronoi partition, one has \( \forall y \in \Gamma, \ y = \mathbb{E} [X|X \in \text{slab}_C(y)] \). Let \( U \) be a finite-dimensional linear subspace of \( H \), \( \Pi_U \) the orthogonal projection onto \( U \) and \( \Gamma = \{ y_1, \ldots, y_N \} \subset U \) a settled point set. Then, the quadratic quantization error with respect to \( \Gamma \subset U \) consist of the projection error and the quantization error of the projected random variable:

\[
\mathcal{E}_N (\Pi_U (X))^2 = \mathbb{E} [X - \Pi_U (X)]^2 + \mathcal{E}_N (\Pi_U (X))^2
\]

Let \( d_N (X) = \min \{ \dim \text{span} (\Gamma) : \Gamma \subset \mathcal{C}_N (X) \} \) denotes the quantization dimension of the level \( N \) of the quantization problem of \( X \). Then it follows that:

\[
\mathcal{E}_N (X)^2 = \mathbb{E} [X - \Pi_V (X)]^2 + \mathcal{E}_N (\Pi_V (X))^2
\]

\( V \subset H \) linear subspace such that \( \dim V \geq d_N (X) \)

Let \( X \) a centered \( H \) valued \( L^2 \) random variable. The covariance operator \( C_X : H \rightarrow H \) of \( X \) is definite by \( C_{XY} = \mathbb{E} [\langle y, X \rangle X] \). In the finite-dimensional case, the matrix of \( C_X \) in the canonical basis is the covariance matrix of \( X \). Particularly, if \( X = (X_i)_{i \in \{0, T\}} \) is a bi-misurable centered process with covariance function \( \Gamma_X (s, t) := \mathbb{E} [X_i X_j] \) satisfying \( \int_{\{0, T\}} \Gamma_X (s, t) \ dt < +\infty \), then \( X \) can be seen as a \( L^2 ([0, T], dt) \) valued random variable with \( \mathbb{E} [X_i^2] < \infty \). Let \( X \) be a centered \( H \) valued random vector with Gaussian distribution \( \mathbb{P}_X, \Gamma \subset H \) a \( N \) stationary codebooks for \( X \) and let \( U = \text{span} (\Gamma) \). Then \( \Pi_U (X) \) and \( X - \Pi_U (X) \) are independent so that \( C_X (U) = U \). That is, the linear subspaces \( U \) of \( H \) spanned by \( N \) stationary codebooks correspond to the principal component of \( X \): in other words, they are spanned by eigenvectors of \( C_X \) corresponding to the \( m \) largest eigenvalues. Thus these subspaces correspond to the first \( m \) principal components of \( X \). At this stage, we note that:

\[
\sum_{j \geq m + 1} \lambda_j^X = \inf \{ \mathbb{E} [\|X - \Pi_V (X)\|^2] : V \subset H \text{ linear subspace, } \dim V = m \}
\]

where \( \lambda_1^X \geq \lambda_2^X \geq \ldots > 0 \) are the ordered non-zero eigenvalues of \( C_X \).

The final representation of \( \mathcal{E}_N (X)^2 \) is:

\[
\mathcal{E}_N (X)^2 = \sum_{j \geq m + 1} \lambda_j^X + \mathcal{E}_N \left( \bigotimes_{j=1}^m \mathcal{N} \left( 0, \lambda_j^X \right) \right)^2
\]

for \( m \geq d_N (X) \)

\[
\mathcal{E}_N (X)^2 < \sum_{j \geq m + 1} \lambda_j^X + \mathcal{E}_N \left( \bigotimes_{j=1}^m \mathcal{N} \left( 0, \lambda_j^X \right) \right)^2
\]

for \( 1 \leq m \leq d_N (X) \)

That is, for the quantization of a Gaussian process \( X \) as soon as we know its Karhunen-Loève basis \( (\mathcal{N}^X_{n})_{n \in \mathbb{N}} \), and its eigenvalues \( (\lambda_n^X)_{n \in \mathbb{N}} \), the problem of optimal \( L^2 \) quantization comes to the problem of the quantization of a Gaussian vector of dimension \( d_N \). Common Gaussian processes have explicit Karhunen-Loève expansion, like the Brownian motion and the Brownian bridge; the Ornstein-Uhlenbeck process admits a semi-closed form for its Karhunen-Loève expansion:

\[
\text{FIGURE 2: A } L^2 \text{ optimal } 1000\text{-quantizer of } (\mathcal{N}(0,1)_t). L^2 \text{ quantization error is equal to } 0.233.\]
Vanilla Option pricing

- Brownian motion \((W_t)_{t \in [0, T]}\):

\[
\begin{align*}
\lambda^W_n (t) & := \sqrt{\frac{2}{T}} \sin \left( \pi \left( n - \frac{1}{2} \right) \frac{t}{T} \right) \\
\lambda^W_n (t) & := \left( \frac{T}{\pi n} \right)^2, \quad n \geq 1
\end{align*}
\]

- Brownian bridge \(B\), on \([0,T]\):

\[
\begin{align*}
\lambda^B_n (t) & := \sqrt{\frac{2}{T}} \sin \left( \pi \frac{t}{T} \right) \\
\lambda^B_n (t) & := \left( \frac{T}{\pi n} \right)^2, \quad n \geq 1
\end{align*}
\]

- Ornstein-Uhlenbeck process \(OU\), on \([0,T]\), starting from 0:

\[
\begin{align*}
\lambda^OU_n (t) & := \left( \frac{1}{\sqrt{\pi T}} \sqrt{\frac{2 \omega^2_{\lambda_n} T}{4 \omega^2_{\beta}}} \right) \sin(\omega_{\lambda_n} t) \\
\lambda^OU_n (t) & := \frac{\sigma^2}{\omega^2_{\lambda_n} + \theta^2}, \quad n \geq 1
\end{align*}
\]

where \(\omega_{\lambda_n}\) are the (sorted) strictly positive solutions of the equation:

\[
\theta \sin (\omega_{\lambda_n} T) + \omega_{\lambda_n} \cos (\omega_{\lambda_n} T) = 0
\]

In the following of the article, the OqM is used to illustrate its performance on simple option pricing cases. One begins with the case of European Plain Vanilla Option Pricing in the Black & Scholes model, for which a closed formula is known and used as a benchmark. The second case consists in pricing an Equity Index Option: its main purposes are to test both from a numerical and graphical point of view the accuracy of OqM and to show the dimensionality effect. Then, in the third case, we show how to use OqM to price path-dependent options like Digital Barrier option.

A one-dimensional case: European Plain Vanilla Option pricing

Let us consider a traded asset \(S\) following a geometric Brownian motion:

\[
dS = \mu S dt + \sigma S dz
\]

where \(\mu\) is the expected instantaneous rate of return on the underlying asset, \(\sigma\) is the instantaneous volatility, and \(z\) is a Wiener process. The volatility and risk-free rate are assumed to be constant throughout the life of the option. It is classical background that, at maturity \(T\):

\[
S(T) = S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma \sqrt{T} \xi}
\]

where \(S_0\) is the asset price at \(t = 0\) and \(\xi\) is a stochastic variable extracted by a Gaussian distribution with zero mean and unit variance. Then one specifies the random variables \(g_x (\xi)\) as follows:

\[
\begin{align*}
g_c (S_T) & = e^{-rT} (S_T - K)_+ \\
g_p (S_T) & = e^{-rT} (K - S_T)_+
\end{align*}
\]

namely, the discounted payoffs of a Call and Put Option, respectively, with strike price \(K\). The formula derived by Black & Scholes can be used to value a European option on a stock that does not pay dividends before the option’s expiration date. Letting \(c\) and \(p\) denote the price at \(t = 0\) of European call and put options, respectively, the formula states that:

\[
\begin{align*}
c & = S_T N (d_1) - Ke^{-rT} N (d_2) \\
p & = Ke^{-rT} N (-d_2) - S_T N (-d_1)
\end{align*}
\]

where

\[
d_1 = \frac{\ln(S_T / K) + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T}
\]

and \(N(\xi)\) is the cumulative normal distribution function. The numerical specifications are as follows: \(S_0 = 100\), \(K = 98\), \(r = 5\%\), \(\sigma = 20\%\), \(T = 2\). At this stage we compute the quantized version of the problem. Precisely, let \(\Gamma = \{y_1, \ldots, y_N\}\) be a \(N\)–optimal quantizer of \(S_T\):

\[
\begin{align*}
\mathbb{E} \left[ g_x (S_T) \right]_{OqM} & = \sum_{i=1}^{N} P_{S_T} \left( C_i (y_i^N) \right) \cdot g \left( y_i^N \right)
\end{align*}
\]

where \(P_{S_T} (C_i (y_i^N))\) is the weight of the Voronoi cell \(C_i\) of the element \(i\) (\(x = c\) for a call option and \(x = p\) for a put option)\(^{17}\). Finally, we compare the resulting quantization error with the Monte Carlo estimator:

\[
\begin{align*}
\mathbb{E} \left[ g_x (S_T) \right]_{MC} & = \frac{1}{N} \sum_{k=1}^{N} g \left( S_{Tk} \right)
\end{align*}
\]

where \(S_{Tk} \sim \mathcal{N}(0; 1)\) and \(N\) is the number of Monte Carlo simulations. The comparison is carried out as follows: we compute a proxy of the standard deviation \(\sigma \left( g_x \right)_{MC}\) of the above estimator using \(M = 10\ 000\) Monte Carlo simulations and we compare it with the quantization error. Some of the numerical values of our tests are reported in the

\(^{17}\)Quantization grids can be downloaded on the website devoted to quantization: www.quantize.maths-fi.com.
Table 1 for Call Option and Table 2 for Put Option. In the first column of the Tables are displayed the number of optimized quantizers used in OqM tests; in the second column are displayed the Black & Scholes option price used as a benchmark, according to (5) and (6); in the third column, the quantized values computed according to (7) and in the fourth column the relative errors with respect to Black & Scholes price; then, in the two last columns, we have written down a proxy of the standard deviation of Monte Carlo estimator and the ratio that measures the error induced by quantization in the scale of the Monte Carlo standard deviation.

A multi-dimensional case: Equity Index Option pricing

What is done in the previous section can be easily generalized to multi-dimensional processes. We have tested OqM to pricing an Equity Index Call Option. The test function is borrowed from classical option pricing in mathematical finance. One considers $d$ traded assets $S_1, \cdots, S_d$ following a $d$-dimensional Black & Scholes dynamics; we assume that these assets are independent and share the same rate of return and volatility, that is: $c_{i} = \cdots c_{d} = \sigma$ and $r_{1} = \cdots r_{d} = r$. At maturity $T$, we then have:

$$S_i(T) = S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma \sqrt{T} z_i} \quad i = 1, \cdots, d$$

We consider an Index $I_d$ (in dimension $d \geq 2$) consist of $n_1$ stocks of $S_1$, $n_2$ stocks of $S_2$ and so on. If, at $t = 0$, $I_0$ is the Index value and $p_i = \frac{n_i S_i(t_0)}{I_0}$ ($0 < p_i < 1$) is the weight of each asset on the Index, the Index value at $t = T$ is:

$$I_T = I (t = T) = I_0 \sum_{i=1}^{d} p_i \frac{S_i(T)}{S_i(I_0)}$$

Then one specifies the random variables $g_c(x)$ as follows:

$$g_c(I_T) = e^{rT}(I_T - K)^+ \quad (8)$$

namely, the discounted payoff of an Equity Index Call Option, with strike $K$. In Figure 3 and Figure 4 are drawn the graphs of $N \rightarrow$ Absolute Error in a log-log scale for function (8), where $N$ are quantizers or Monte Carlo simulations. Precisely, the continuous blue line shows Monte Carlo Absolute Error $\log (N) \rightarrow -\frac{1}{2} \log (N) + \log \sigma g_c(x_N)$. We assume that “exact value” of Index, $I_T$, and then Call option price are obtained by 100 000 Monte Carlo simulations. The function $f = \frac{1}{\sqrt{N}}$ is showed by a dashed red line, provided as comparison. Finally, $\cdot$ plot displays OqM Absolute Error (with respect to previous “exact value” of Index Call option price) concerning some values of $N$-quantizers. The green straight line is obtained by Lagrange’s polynomial interpolation, first degree. The test is processed in dimension $d \geq 2$. One observes that slop of Monte Carlo Absolute Error line is obviously the same as function $f = \frac{1}{\sqrt{N}}$ and the OqM Absolute Error line follows the same trend for $d = 4$. Therefore, the slop of OqM Absolute Error line should $2/d$ ($d = 4$) according to (2) and (3); theoretical rate for the error bounds is confirmed. Moreover, when $d \leq 4$ quantization over performs more than Monte Carlo method as $N$ increases. When $d > 4$, this is at most true up to a critical number of points, $N_{\text{critical}}$. One verifies that, e.g., OqM behaves better than Monte Carlo method in dimension $d = 6$ as long as $N$ is lower than about $N_{d=6}^{\text{critical}} = 500$.

A multi-dimensional and path-dependent case: Multi-asset Digital Barrier Option pricing

In this section, we assume that $S_1, \cdots, S_d$ are standard Brownian motions on $[0, T]$. We are interested in the value of risk-neutral expectation of a path-dependent payoff of a diffusion based on $S_1, \cdots, S_d$, namely $E[F(S_{t_1}, \cdots, S_{t_n}, \cdots, S_{t_{4}}, \cdots, S_{t_{4}})]$, where $0 \leq t_1 < \cdots < t_n \leq T$ are $n$ dates of interest for the underlying processes. Let assume $F$ be a Digital Barrier Option for which payoff depends on whether or not the ratio $\frac{S_{t_j}(I)}{S_{t_j}(I_0)} \forall j = 1, \cdots, d$, $\forall i = 1, \cdots, n$ touched a barrier level $B$ (in percentage) at some time $0 \leq t_1 < \cdots < t_n \leq T$ during the life of the option. The value of the payoff is not affected by the size of the difference between the underlying and a strike price, but it is in the form of a constant cash payment $P$. Then, we assume:

$$\text{Payoff Digital Down & Out} = \begin{cases} P & \text{if } \frac{S_{t_j}(I)}{S_{t_j}(I_0)} > B \\ 0 & \text{otherwise} \end{cases} \quad \forall j = 1, \cdots, d \text{ and } \forall i = 1, \cdots, n.$$

The option described here are multi-dimensional and path-dependent, which means that the payoff profile depends on the $n$ values at $0 \leq t_1 < \cdots < t_n \leq T$ of $d \geq 2$ traded assets. We have tested OqM to price this option in dimension $d = 2$. The numerical specifications are as follows:

$S_1(t_0) = 2.5, S_2(t_0) = 8, B = 0.8, r = 5\%,$

$\sigma_1 = 10\%, \sigma_2 = 20\%, P = 100, T = 1, \text{time steps} = 6$

The numerical results are given in Table 3.

Winter 2014
FIGURE 3: Convergence rate of absolute error for Optimal quantization Method versus Monte Carlo simulations in dimension $d=3$ on the left and $d=4$ on the right.

FIGURE 4: Convergence rate of absolute error for Optimal quantization Method versus Monte Carlo simulations in dimension $d=5$ on the left and $d=6$ on the right.

Call option

<table>
<thead>
<tr>
<th>N-optimized quantizer</th>
<th>Black&amp;Scholes Call option reference value</th>
<th>OqM value</th>
<th>Relative error (%)</th>
<th>MC Standard Deviation [n. Sim = 10 000]</th>
<th>Absolute error / MC St. Dev</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>17.2086</td>
<td>17.2018</td>
<td>3.9797E-04</td>
<td>23.5499</td>
<td>2.9181E-04</td>
</tr>
<tr>
<td>100</td>
<td>17.2086</td>
<td>17.2060</td>
<td>1.4929E-04</td>
<td>23.5499</td>
<td>1.0909E-04</td>
</tr>
<tr>
<td>200</td>
<td>17.2086</td>
<td>17.2084</td>
<td>1.3214E-05</td>
<td>23.5499</td>
<td>9.6557E-06</td>
</tr>
<tr>
<td>500</td>
<td>17.2086</td>
<td>17.2085</td>
<td>5.8687E-06</td>
<td>23.5499</td>
<td>4.2885E-06</td>
</tr>
<tr>
<td>1000</td>
<td>17.2086</td>
<td>17.2086</td>
<td>5.506E-07</td>
<td>23.5499</td>
<td>4.02E-07</td>
</tr>
</tbody>
</table>

TABLE 1: European Plain Vanilla Call Option values with respect to the Black & Scholes closed formula.

Put option

<table>
<thead>
<tr>
<th>N-optimized quantizer</th>
<th>Black&amp;Scholes Call option reference value</th>
<th>OqM value</th>
<th>Relative error (%)</th>
<th>MC Standard Deviation [n. Sim = 10 000]</th>
<th>Absolute error / MC St. Dev</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5.8827</td>
<td>5.8385</td>
<td>7.5107E-04</td>
<td>9.8691</td>
<td>4.4769E-03</td>
</tr>
<tr>
<td>50</td>
<td>5.8827</td>
<td>5.8803</td>
<td>3.9819E-04</td>
<td>9.8691</td>
<td>2.3735E-04</td>
</tr>
<tr>
<td>100</td>
<td>5.8827</td>
<td>5.8813</td>
<td>2.4098E-04</td>
<td>9.8691</td>
<td>1.4364E-04</td>
</tr>
<tr>
<td>200</td>
<td>5.8827</td>
<td>5.8827</td>
<td>1.0843E-05</td>
<td>9.8691</td>
<td>6.4635E-06</td>
</tr>
<tr>
<td>500</td>
<td>5.8827</td>
<td>5.8826</td>
<td>9.1915E-06</td>
<td>9.8691</td>
<td>5.4788E-06</td>
</tr>
<tr>
<td>1000</td>
<td>5.8827</td>
<td>5.8827</td>
<td>3.8830E-07</td>
<td>9.8691</td>
<td>2.314E-07</td>
</tr>
</tbody>
</table>

TABLE 2: European Plain Vanilla Put Option values with respect to the Black & Scholes closed formula.
TABLE 3: Digital Down & Out Option values.

<table>
<thead>
<tr>
<th>fixing dates</th>
<th>OqM value (N_quantizer = 1900)</th>
<th>Monte Carlo (N_simulations = 10 000)</th>
<th>Relative error (%)</th>
<th>[(MC-OqM)/MC]</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>82.6837</td>
<td>82.0727</td>
<td>7.4446E-03</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>82.5263</td>
<td>81.5198</td>
<td>1.2347E-02</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>82.0455</td>
<td>80.9515</td>
<td>1.3514E-02</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>82.3328</td>
<td>80.4887</td>
<td>2.2911E-02</td>
<td></td>
</tr>
</tbody>
</table>

About the Author

Gaetano Marino Financial Consultant at Iason ltd.
Email address: gaetano.marino@iasonltd.com

About the Article

Submitted: February 2013.
Accepted: April 2013.

References


Concentric is a company specialized in the advisory, research and training in sales and finance. The company adopts a dynamic approach to its organizational evolution, reflecting ongoing changes in the business environment, the requirements of clients and developments in best practices.

QUANTITATIVE ADVISORY
quad is the fusion of Concentric and iason international risk advisory, research and training practitioner teams, rigorous project methodologies and tried-and-tested tools.

CORPORATE GOVERNANCE ADVISORY
cga comprises organisational analysis, design and development projects for bank, insurance company and corporate compliance.

CONCENTRIC RESEARCH
core includes service consultation, needs analysis, project modeling, satisfaction analysis, industry survey and financial analysis.

FINANCE MASTER CLASS
fmc is one of Europe’s most appreciated and successful public technical financial training programs for banking and insurance professionals.

SALES AND FINANCE EDUCATION
the safe team adopts a holistic and value-adding approach to the design and development of in-company finance, commercial and management training projects.

For more information visit www.concentric-italy.com
In this section Antonio Castagna, co-founder and C.E.O. of Iason Ltd, interviews Fabio Mercurio of Bloomberg, L.P.. Fabio Mercurio is very well known in the financial industry as one of the major experts in quantitative modelling, and interest rate derivatives in particular. Fabio has published extensively in international journals and magazines, including 13 cutting-edge articles in Risk, and has co-authored, jointly with Damiano Brigo, the celebrated, and by now classic, book “Interest Rate Models – Theory and Practice”. Currently, Fabio is head of Derivatives Research at Bloomberg, L.P. He is also adjunct professor at NYU, a member of CME risk committee and the president of the Scientific Committee of Iason Ltd.

A.: Hello everybody. This is Antonio Castagna and we are here today with Fabio Mercurio of Bloomberg, L.P.: Hi Fabio!

F.: Hi Antonio!

A.: I had the pleasure to work with Fabio in Banca IMI, which we both joined in late 90’s, Fabio as a quant and I as an option trader. Now I have the pleasure of interviewing him: thank you, Fabio, for accepting our invitation to contribute to the inaugural issue of Argo, not only with this interview, but also by submitting an article on credit risk.

F.: Thank you, Antonio, for your very nice introduction. Needless to say, the pleasure is mine as well.

A.: Great, let’s start then. Using, admittedly rather loosely, the philosopher Vico’s historic categories, after the divine period spanning from the ’60s to the mid ’80s, when the fundamentals of the asset (and derivative) pricing theory were laid down; and after the heroic period from the second half of the ’80’s to year 2006, when the main theory was refined and then extended (stochastic volatility, sophisticated interest rate modelling, etc.); I would say that we entered, in 2007, the human period, which honestly I cannot fully define or describe. Do you agree on this rough tri-partition of the development of finance and financial evaluation theory, and how would you define the period we are currently experiencing?

F.: It is an interesting view, Antonio, but let’s put it this way: I think there are “sparkles of divine lights” and “traces of heroic efforts” also in the more recent Mathematical Finance production. It is just too early to make a fair comparison and issue a final verdict. It is true that quants have been struggling to stand up on their feet again, after been so heavily punched by the financial crisis. But, we just took a temporary break to better study and analyse this unprecedented and fascinating reality. We will come back more aware and stronger than before.

A.: Well, from your answer it seems we are in a transition period where things are not completely settled down, and this somehow confirms Vico’s categories. I think the passages through different phases are also related to the fact that Mathematical Finance started in Economic Departments, and then it became a branch of Mathematics. It seems that gradually those studying the problem to model financial risk factors and to evaluate financial contracts, disregarded more and more the economic background. Don’t you think that the current environment requires a deeper understanding of the micro- (and macro-) economic mechanics driving financial markets?

F.: Antonio, this is clearly a rhetorical question, which is also very general. Applied scientists should not lose focus on the true scope of their research and should resist the seductive temptation of drifting to more theoretical matters. Theory and practice present different challenges. But some researchers sometimes find it more comforting to hide behind implausible assumptions and fancy theoretical constructions than facing risky, and potentially frustrating, reality checks.
A.: Popper used to call them ad-hoc assumptions, to save a theory from counterfactual evidence.

F.: Exactly. At the same time, however, we cannot impose any preferred direction to follow. Not only because individuals should be free to express themselves the way they please, but also because it would be a capital mistake to believe that knowledge can only be achieved by following predefined rules and paths. The quality of our lives is often built on practical accomplishments made also possible by the visionary work of theoretical scientists. So, to conclude, I agree we should have a deeper understanding of economic principles and drivers, but the path leading to knowledge and great discoveries, also in Mathematical Finance, is often unpredictable. A theoretical detour on a secondary road may easily lead to a highway of unparalleled practical applications.

A.: Ok, let’s then hope these detours do not lead into dead end streets. But generally speaking, don’t you think that Quantitative Finance, as we knew until 2006, is over? I have to confess that attending quant conferences nowadays can be quite disappointing.

F.: Quantitative finance is in constant evolution. For instance, now everybody feels comfortable with the idea of dealing with implied volatility smiles. But they came up as a shocking reality only during the 1989 stock market crash. It always happened that older models have been surpassed by newer ones designed to accommodate unprecedented market features. So, in this sense, Quant Finance after the credit crunch is no exception. It just seems worse because of the aftermath of the financial crisis, which ended up hitting quants as well. Of course, a lesson we learned is that too much Mathematical modelling can be dangerous, but too little is clearly not good either. We just need to shift focus and adapt to the new world. As per your comment on quant conferences, I think you are being too harsh.

A.: Well, you know that my greatest, but by no means worst, flaw is that I take very clear-cut views...

F.: Ah, that’s true! Anyway, you may be right saying that some research works are based on questionable assumptions. But I would still regard them as a progress. The truth is that it always took a long time to make big steps forward. For instance, how do you justify it roughly took ten years to add a simple time dependence to the parameters in Vasicek’s model? It may take an hour to turn a page and learn a new model on a textbook. But innovation lies on a much wider time scale.

A.: Moving to the sell-side, which is related to the Quantitative Finance development in a sort of circular effect mechanism, at the moment it looks like banks are focussing to make market on simple contracts for their clients. Do you think there will be room in a near future for a revamp of complex pay-offs and structures?

F.: Yes, it is true that the fancy structured notes quants had so much fun with are less and less popular. But we must also stress the shift in complexity that occurred after the 2007 credit crunch. In the current market environment even a simple deal such as a single-currency interest-rate swap may be complex and hard to value. Modelling and valuing, if any, collateral features, funding and liquidity costs, counterparty credit risk along with other possible sources of cost and risk is by no means less complex than pricing an “old” callable path-dependent multi-currency and multi-asset derivative. The problem is that some of these concepts are new to a wider audience, and unfortunately still largely undefined. What’s the point of modelling with Mathematical rigor quantities that are imprecisely defined? So, to summarize, there is still a great deal of complexity out there spicing up our quant lives. But it is a different type of complexity, even greater than before at epistemological level!

A.: I could not agree more with you and I would love to discuss about it for hours, but I think that it is time to end our conversation. Thank you, Fabio, for being with us and for sharing with us your views.

F.: Thank you so much Antonio for inviting me here. It was a pleasure to be interviewed by you. I wish you all the best, and good luck with Argo!
Energy & Commodity Finance

- Energy Option Pricing
- Quantitative Modelling
- Commodity Trading
The research and consulting network in energy and commodity finance

Argo Review
The digital quarterly magazine on best practices and models for bank and corporate risk management.
Check latest issues>

Garp ERP®
Full training in preparation to the Garp Energy Risk Professional (ERP) certification exam.
See GARP ERP training details>

Codes
A basket of computer codes for efficient pricing, trading, and managing energy and commodity linked positions.
Download computer codes>

Articles
A continuous flow of papers concerning, developing, and explaining cutting-edge techniques in commodity finance.
Follow our publications>

Books
Comprehensive monographs encompassing key topics in physical and financial risk management.
See more about our volumes>

Programs
Academic programs offering high-level knowledge and skills in energy and commodity finance.
Check academic programs list>

FloRisk™
A web-based system of advanced solutions for pricing, monitoring, and managing energy and commodity risk.
Get more about our solution>

The New Frontier of Cloud Consulting

Free registration

© All rights reserved
Follow us on www.energisk.org or contact us at contact@energisk.org
In this article Marena, Roncoroni, and Fusai derive a closed-form formula for the fair value of call and put options written on the arithmetic average of security prices driven by jump diffusion processes displaying (possibly periodical) trend, time varying volatility, and mean reversion. The model allows one for jointly fitting quoted futures curve and the time structure of spot price volatility. These result extends the no-jump case put forward in [Fusai, G., Marena, M., Roncoroni, A. 2008. Analytical Pricing of Discretely Monitored Asian-Style Options: Theory and Application to Commodity Markets. Journal of Banking and Finance 32 (10), 2033-2045]. A few tests based on commodity price data assess the importance of introducing a jump component on the resulting option prices.

In Fusai, Marena, and Roncoroni (2008), we put forward a procedure for pricing Asian-style options under the following assumptions:

- Underlying spot price dynamics are driven by continuous diffusion processes, say $(S_t)_{t \geq 0}$, possibly exhibiting mean reversion, time varying trend, and time varying volatility.
- Asian-style options are either puts or calls, including the cases of fixed as well as floating (i.e., depending on the underlying asset quote) strike price.

Under these hypotheses, we devised a new method to calculate the exact analytical expression for the moment generating function of the joint pair consisting of commodity spot price $S_{n\Delta}$ computed at the option’s maturity $T := n\Delta$ and the weighed arithmetic average $\sum_{i=0}^{n\Delta} \alpha_i S_i$ over the option lifetime. That result allowed us to derive analytical expressions for the relevant transforms of option prices with respect to the strike price. Finally, using the Fourier inversion method, we got to semi-analytical expressions for a large class of Asian-style derivatives.

To the best of our knowledge, the resulting prices are the sole closed-form formulae for options written on arithmetic averages (up to inverse transformation). This comes as opposed to the large number of numerical approximations proposed in the financial literature for those securities.

This article aims at extending our previous result to the case of spot price dynamics driven by a 2-factor jump-diffusion process. We manage to preserve the ability of the model to reproduce mean reversion, time varying trend, as well as time varying volatility. However, we allow for the underlying variable to exhibit discontinuous paths, as is often the case in several financial markets, in particular for energy sources and other commodities.

The rest of the article is organized as follows. The first part states the problem; then there is the setting of the model and the computation of the relevant moment generating function. The result-
We consider a time horizon $[0, T]$ representing the option’s lifetime: $0$ is the valuation date, while $T$ is the time of expiration. At time $T$, the option pays out an amount that is contingent upon the realization of a price average $\text{Avg}_n := \sum_{j=0}^{n} \frac{1}{T} S_{j\Delta}$ (with $\sum a_j = 1$) of discretely monitored spot prices $S_0, S_{\Delta}, ..., S_{n\Delta}$. Specifically, we consider pay-off functions of Asian-style options under discrete monitoring, as illustrated in the section after and its followed by a sensitivity analysis of our formulae; conclusions provide with a few indications about directions for future research on the subject.

**Statement of the Problem**

We consider a time horizon $[0, T]$ representing the option’s lifetime: $0$ is the valuation date, while $T$ is the time of expiration. At time $T$, the option pays out an amount that is contingent upon the realization of a price average $\text{Avg}_n := \sum_{j=0}^{n} \frac{1}{T} S_{j\Delta}$ (with $\sum a_j = 1$) of discretely monitored spot prices $S_0, S_{\Delta}, ..., S_{n\Delta}$. Specifically, we consider pay-off functions of Asian-style options under discrete monitoring, as illustrated in the section after and its followed by a sensitivity analysis of our formulae; conclusions provide with a few indications about directions for future research on the subject.

**TABLE 4: Payoff functions of Asian-style options under discrete monitoring.**

<table>
<thead>
<tr>
<th>Option</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed Strike</td>
<td>$\max \text{Avg}_n - k, 0$</td>
</tr>
<tr>
<td>Floating Strike</td>
<td>$\max S_{n\Delta} - \text{Avg}_n - k, 0$</td>
</tr>
</tbody>
</table>

**Step 1.** Compute the moment generating function (MGF) of the underlying spot price $S$ at maturity $T = n\Delta$ conditional to $S_0 = s_0$ at current time $0$:

$$\gamma \rightarrow \nu_{0,s_0}(\gamma) := E_0 \left[ e^{-\gamma S_{n\Delta}} \right]$$

**Step 2.** Using the main theorem in Fusai, Marena, and Roncoroni (2008), calculate the MGF of the pair $(S_{n\Delta}, \sum_{j=0}^{n} a_j S_{j\Delta})$ by recursion:

$$(\gamma, \mu) \rightarrow \nu_{0,s_0}^{n\Delta}(\gamma, \mu) := E_0 \left[ e^{-\gamma S_{n\Delta} + \sum_{j=0}^{n} \gamma S_{j\Delta}} \right]$$

Notice that $\nu_{0,0}(\gamma) = \nu_{0,0}^{n\Delta}(\gamma, 0)$.

**Step 3.** Consider a contingent claim paying off $(Y_T - k)^+$ at time $T$, where $k$ is the strike and $Y$ is a nonnegative Markovian stochastic process. This form includes plain vanilla calls $(Y_T = S_{n\Delta})$ and standard fixed strike Asian-style options $(Y_T = \sum_{j=0}^{n} a_j S_{j\Delta})$ struck at $k$. The time $0$ arbitrage-free option price seen as a function of the strike price $k$ reads as:

$$k \rightarrow C_{0,0}^T(k) = e^{-rT} \int_0^{\infty} f_{Y_T}^k(y-k) f_{Y_T}(y) \, dy$$

where $f_{Y_T}$ denotes the risk-neutral probability density of $Y_T$ conditional to $Y_0 = y_0$. Provided that the MGF of $Y_T$ exists, define the Laplace transform $L$ of the option price $C_{0,0}^T(k)$ with respect to the strike price $k$ as:

$$\lambda \rightarrow L[C_{0,0}^T(.) | (\lambda)] := \int_0^{\infty} e^{-\lambda k} C_{0,0}^T(k) \, dk$$

$$= e^{-rT} \left( E_0 \left[ e^{-\lambda Y_T} \right] + \frac{E_0(Y_T) - 1}{\lambda} \right)$$

The option price can be written as:

$$C_{0,0}^T(k) = e^{-rT} \left( L^{-1} \left[ E_0 \left[ e^{-\lambda Y_T} \right] \right](k) + E_0(Y_T) - k \right)$$

where:

- expected values $E_0 \left[ e^{-\lambda Y_T} \right]$ and $E_0(Y_T)$ can be computed based on the output at step 2; whilst $E_0(Y_T) = \sum_{j=0}^{n} a_j E_0(S_{j\Delta}) = \sum_{j=0}^{n} a_j F(0, j\Delta)$, where $F(0, j\Delta)$ is the forward price for maturity $j\Delta$

- transform inversion can be executed using the Fourier inversion method (see Fusai and Roncoroni (2008)).
We refer to a table reported in Fusai, Marena, and Roncoroni (2008) for exact expressions across the variety of cases under concern.

An appropriate selection of $\gamma$, $\mu$, and $s_i$ allows one to cover the cases of standard European, fixed strike Asian-style, fixed strike volume weighted Asian-style, and floating strike standard Asian-style options.

The goal of this article is to tune this procedure in a way to encompass the case of underlying prices driven by a jump-diffusion process. The key point here is that steps 1 to 3 stated earlier do occur in automatic cascade, meaning that each step directly follows from the previous one. Consequently, we just need to show that step 1 delivers closed-form output under jump dynamics for the spot price and the rest follows with no particular change.

**Model Setting**

We consider price dynamics under a risk-neutral reversion to (possibly) time varying trend, time varying volatility, and a jump component of Poisson type, leading to a general expression:

$$dS_t = \beta (\eta_t - S_t) dt + \sigma_s \sqrt{S_t} dW_t + dJ_t,$$

(9)

where:

- $\beta$ is a mean reversion constant frequency expressed in $1/time$ units;
- $(\eta_t)_{t \geq 0}$ is a deterministic time-varying price trend which spot quotes revert to;
- $(\sigma_t)_{t \geq 0}$ is a deterministic time-varying spot price volatility function: squares $\sigma_t^2$ represents the time $t$ variance of instantaneous price variations per unit of price value $S_t$ and is expressed in $1/time$ units;
- $(W_t)_{t \geq 0}$ is a standard Brownian motion;
- $(J_t)_{t \geq 0}$ is a compound Poisson process $\sum_{i=1}^{N_t} Y_i$ with the following properties:
  - $N_t$, $Y_i$’s, and $W_t$ are all mutually stochastically independent;
  - Jump intensity $\lambda_t$ is deterministic, time varying, and bounded by a constant from above;
  - Jump magnitudes $Y_i$ are i.i.d. copies of an exponential variable $Y$ with mean $\xi > 0$;
- $(\mathcal{I}_t)_{t \geq 0}$ is the compensated martingale process defined as: $d\mathcal{I}_t := dJ_t - \xi\lambda_t dt$.

Notice that drift components $\beta$ and $\eta$ are meant under a risk-neutral probability: in principle they cannot be statistically estimated on time series of observed spot prices, but they ought to be calibrated on plain vanilla option quotes via pricing formulae as those we have described in the previous section. In particular, drift term $\eta_t$ can be selected in a way that model (9) fits an observed forward price curve $(F_{0,T}, T \geq 0)$ quoted in the market, i.e.,

$$E_0^* (S_T) = F_{0,T},$$

(10)

where the $^*$ superscript indicate that expectation is computed under $\mathbb{P}^*$. By inserting the integral version of dynamics (9) into this formula, we come up to identifying the risk-neutral trend function:

$$\eta_T = F_{0,T} + \frac{1}{\beta} \partial_T F_{0,T}$$

(11)

ensuring the claimed fitting of the observed forward curve.

**Spot Price MGF**

We now compute an analytical expression for the MGF of the underlying spot price $S_{t+\Delta}$, conditional to the market information available at time $t$, which is formally represented by the $\sigma$-algebra $\mathcal{F}_t^S$ generated by the price process $(S_t)_{t \geq 0}$ up to time $t \in [0,T]$.

Under spot price dynamics (9), the MGF of the spot price $S_{t+\Delta}$ given $S_t = x$ is:

$$M_{t,x} (\gamma) = e^{-[A_t(\Delta\gamma)x + B_t(\Delta\gamma)]}$$

where:

$$A_t(\Delta; \gamma) = \frac{\gamma e^{-\beta\Delta}}{1 + \frac{\gamma}{2} \int_t^{t+\Delta} e^{-\beta(t+\Delta-u)} du},$$

(12)

$$B_t(\Delta; \gamma) = \gamma F_{0,t+\Delta} - F_{0,t} A_t(\Delta; \gamma) + \frac{1}{2} \int_t^{t+\Delta} F_{0,u}\sigma_u^2 A_u(\Delta; \gamma)^2 du + \frac{\xi^2}{2} \int_t^{t+\Delta} \lambda_u \left( \frac{A_u(\Delta; \gamma)^2}{1 + \xi A_u(\Delta; \gamma)} \right) du$$

(13)

**Proof.** Consider the MGF $M_{t,x} (\gamma^\ast)$ as a function $v(t,x)$ for fixed $\Delta$ and $\gamma$. Similarly, define $A(t)$ and $B(t)$ as $A_t(\Delta; \gamma)$ and, respectively, $B_t(\Delta; \gamma)$. Function $v$ solves the partial integro-differential equation:

$$\partial_t v (u,x) + [\beta (\eta_u - x) - \lambda_u \xi] \partial_x v (u,x) + \frac{1}{2} \sigma_u^2 \partial^2_{xx} v (u,x) + \lambda_u E_u [v (u, x + Y) - v (u,x)] = 0$$

$$v(t+\Delta, x) = e^{-\gamma x}$$

$$v(t,x) = e^{-\gamma x} M_{t,x} (\gamma^\ast),$$

(14)

$$M_{t,x} (\gamma^\ast) = e^{-[A(t) (\Delta; \gamma^\ast)x + B(t) (\Delta; \gamma^\ast)]}$$

(15)
on \([t, t + \Delta] \times \mathbb{R}\).

We consider a solution with exponential affine structure:

\[
v(t, x) = e^{-A(t)x - B(t)}
\]

which would lead to the following ODE system for functions \(A(t)\) and \(B(t)\):

\[
\begin{aligned}
-A'(t) + \beta A(t) + \frac{1}{2} \sigma^2 A(t)^2 &= 0 \\
B'(t) - \beta \eta - \xi A(t) - \lambda t \xi A(t) &= 0
\end{aligned}
\]

with boundary conditions \(A(t + \Delta) = \gamma\) and \(B(t + \Delta) = 0\).

Let \(C(t)\) be defined by:

\[
A(t) = e^{\beta t} C(t)
\]

Plugging this expression into the relevant equation, we have:

\[
\begin{aligned}
- \left( \beta e^{\beta t} C(t) + e^{\beta t} \partial_t C(t) \right) + \\
+ \beta e^{\beta t} C(t) + \frac{1}{2} \sigma^2 e^{2\beta t} C(t)^2 &= 0
\end{aligned}
\]

\[
C(t + \Delta) = \gamma e^{-\beta(t + \Delta)}
\]

By separating variables, we get:

\[
C(t) = \frac{\gamma e^{-\beta(t + \Delta)}}{1 + \frac{\beta}{2} \int_t^{t+\Delta} e^{\beta(T - u)} du}
\]

which, combined with assumption (14), leads to expression (12).

From the second ODE, we have

\[
B(t) = \beta \int_t^{t+\Delta} \eta \lambda \xi A(u) du + \\
+ \xi \left( - \int_t^{t+\Delta} \lambda \xi A(u) du + \int_t^{t+\Delta} \lambda \xi A(u) \right)
\]

By using (11), we get expression (13).

**Remark.** In absence of jumps, the stated Proposition matches Lemma 5 in Fusai, Marena, and Roncoroni (2008).

**Pricing Algorithm**

Combining the result obtained in the previous section with the procedure described earlier, we come up to the following algorithm for pricing Asian-style call options:

**Algorithm**

- **Step 0:** Assume:
  - A time interval \([0, T]\), which refines into \(n\) monitoring lags of length \(\Delta\), and a strike index \(k\);
  - A continuously compounded rate of interest \(r\);
  - Risk-neutral spot dynamics:
    - mean reversion freq. \(\beta\)
    - fwd curve \((F_{0, \lambda})_{0 \leq \lambda \leq T}\)
    - volatility \((\eta_{0, \lambda})_{0 \leq \lambda \leq T}\)
    - jump freq. \((\lambda_{0, \lambda})_{0 \leq \lambda \leq T}\)
    - average size of jump \(\xi\)
    - starting price \(S_0\)

- **Step 1.** Compute the MGF of \(S_{(j+1)\Delta}\) conditional to \(S_{j\Delta} = x\), for \(j = n - 1, n - 2, ..., 0\), using formula:

  \[
  \gamma \to v_{j\Delta, x} \left( \gamma \right) = e^{-\left[ A_{j\Delta}(\gamma) x + B_{j\Delta}(\gamma) \right]}
  \]

  with:

  \[
  A_{j\Delta}(\gamma) = \frac{\gamma e^{-\beta \Delta \xi}}{1 + \frac{\beta}{2} \int_j^{j+\Delta} e^{\beta(T - u)} du}
  \]

  \[
  B_{j\Delta}(\gamma) = \gamma f_{0, 0} - f_{0, 0} A_{j\Delta}(\gamma) - \frac{1}{2} \int_j^{j+\Delta} f_{0, 0} e_\Delta^2 A_{j\Delta}(\gamma) du + \frac{\sigma^2}{2} \int_j^{j+\Delta} \lambda_{0, \gamma} A_{j\Delta}(\gamma) du
  \]

- **Step 2.** Compute the MGF of the pair \(\left(S_{n\Delta}, \sum_{j=0}^{n-1} \eta_{j\Delta, \gamma} S_{j\Delta}\right)\) conditional to \(S(0) = S_0\):

  \[
  (\gamma, \mu) \to v_{0, \Delta, x} \left( \gamma, \mu \right) = e^{-\left[ A_{0\Delta}(\gamma) x + B_{0\Delta}(\gamma, \mu) \right]}
  \]

  where the function \(\Lambda_j(\gamma; \gamma, \mu)\) satisfies the recursive equation:

  \[
  \Lambda_j(\Delta; \gamma, \mu) = A_{j\Delta}(\Delta; \gamma, \mu) x + B_{j\Delta}(\Delta; \gamma, \mu)
  \]

  for \(j = n - 1, n - 2, ..., 0\), starting with:

  \[
  \Lambda_n(\Delta; \gamma, \mu) = \gamma + \mu \eta
  \]

  Here, \(A_{j\Delta}\) and \(B_{j\Delta}\) are as in Step 1.

- **Step 3.** The fixed-strike Asian-style call option price can be represented as:

  \[
  C_{\Delta, k}^{\text{Asian}}(k) = e^{-rT} \left( L^{-1} \left[ \frac{v_{0, \Delta, x}(0, \mu)}{\nu} \right] (k) + \sum_{j=0}^{n-1} a_j f_{0, j\Delta} - k \right)
  \]

  Whenever the analytical inverse of transform \(L\) is not available, numerical evaluation is
FIGURE 5: Heating Oil futures price curve fitting October 31, 2010, market quotes (panel 1); price jump frequency function (panel 2).

FIGURE 6: Asian-style options price against mean reversion frequency level and varying assumptions about the jump frequency function.

FIGURE 7: Asian-style options price against spot price volatility parameter and varying assumptions about the jump frequency function.
required. For instance, the Fourier-Euler algorithm proposed in Abate and Whitt (1992) leads to pricing formula (15) with:

$$L^{-1}\left[\frac{v_{0,\Delta}(0,\mu)}{\rho^2}\right](k)\approx \sum_{m=0}^{M} (\frac{\mu}{2\pi})^{2m} d_{N+m}(k)$$

with:

$$d_{\rho}(k) = \frac{\rho^{\mu}}{2\pi} \sum_{j=1}^{\rho} (-1)^{j} \frac{\rho^{\mu}}{\mu^2}$$


\(N\) and \(M\) are suitable constants, and \(a_t\) is located to the right-hand side of the real part of the largest singularity of the Laplace transform, i.e., \(a_t > 0\). We suggest to adopt the following parametric setting: \(a_t = 18.4, M = 25, N = 15\) (see Fusai and Roncoroni (2008) for details).

### Pricing Analysis

We used our formula to evaluate Asian-style call options written on Heating Oil price averages. Our goal is to assess option price sensitivity to key input parameters and data including time-to-maturity, market forward curve, and jump frequency.

We begin by defining values for each of the input quantities indicated on step 0 of the pricing algorithm stated earlier. Our base case assumes that:

- Current time is \(0 := October 31, 2012\).
- Options mature within \(T = 3, 6, 9, \text{ and } 12\ \text{months}\).
- Averages are computed based on monthly monitoring, i.e., \(\Delta = 1/12\ \text{years}\).
- Strike index \(k\) is assumed to match the ATM level defined as:

$$\bar{\Delta}_{0,n} := \mathbb{E}_0^\rho (\Delta_{0,n}) = \frac{1}{n+1} \sum_{j=0}^{n} F_{0,j\Delta}$$

where \(n = T/\Delta\). Table 5 provides these values for the cases under consideration.

- For each maturity, interest rate \(r\) is bootstrapped from LIBOR quotes on spot date 0.
- Mean reversion frequency \(\beta = 0.1 \ \text{per annum}\).
- Heating Oil standing forward curve is reported in Table 7: a continuous curve \(F_0\), obtains through interpolation using a cubic spline over the set of quoted values; this procedure results into the path shown in Figure 5 (panel 1).
- Spot price volatility parameter is constant \(\sigma = 0.7 \ \text{per annum}\).
- Jump frequency is indicated in Table 6, where February experiences the highest value of the calendar year: a continuous curve \(\lambda\) obtains through linear interpolation over the set of assigned values; this procedure results into the curve shown in Figure 5 (panel 2).
- Average size of jump \(\xi = 0.1 \times \bar{\Delta}_{0,n}\), that is 10% of the average expected spot price. Table 5 provides these parameters across varying maturities.

- Current spot price is set equal to \(s_0 := F_{0,0}\).

We now build five alternative assessment of the jump frequency:

### Table 6: Time varying jump frequency.

<table>
<thead>
<tr>
<th>Time</th>
<th>Jump Intensity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1m</td>
<td>5.00</td>
</tr>
<tr>
<td>2m</td>
<td>5.50</td>
</tr>
<tr>
<td>3m</td>
<td>6.00</td>
</tr>
<tr>
<td>4m</td>
<td>5.50</td>
</tr>
<tr>
<td>5m</td>
<td>5.00</td>
</tr>
<tr>
<td>6m</td>
<td>4.50</td>
</tr>
<tr>
<td>7m</td>
<td>4.00</td>
</tr>
<tr>
<td>8m</td>
<td>3.50</td>
</tr>
<tr>
<td>9m</td>
<td>3.00</td>
</tr>
<tr>
<td>10m</td>
<td>3.50</td>
</tr>
<tr>
<td>11m</td>
<td>4.00</td>
</tr>
<tr>
<td>12m</td>
<td>4.50</td>
</tr>
</tbody>
</table>
FIGURE 8: Asian-style options price against jump size mean and varying assumptions about the jump frequency function.

FIGURE 9: Asian-style options price against moneyness and varying assumptions about the jump frequency function.

FIGURE 10: Asian-style options price against maturity and varying assumptions about the jump frequency function.
<table>
<thead>
<tr>
<th>Delivery Month</th>
<th>Maturity</th>
<th>Settlement Price (USD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>December</td>
<td>30-Nov-2012</td>
<td>3.0682</td>
</tr>
<tr>
<td>January</td>
<td>31-Dec-2012</td>
<td>3.0623</td>
</tr>
<tr>
<td>February</td>
<td>31-Jan-2013</td>
<td>3.0519</td>
</tr>
<tr>
<td>March</td>
<td>28-Feb-2013</td>
<td>3.0400</td>
</tr>
<tr>
<td>April</td>
<td>29-Mar-2013</td>
<td>3.0100</td>
</tr>
<tr>
<td>May</td>
<td>30-Apr-2013</td>
<td>2.9800</td>
</tr>
<tr>
<td>June</td>
<td>31-May-2013</td>
<td>2.9900</td>
</tr>
<tr>
<td>July</td>
<td>28-Jun-2013</td>
<td>2.9687</td>
</tr>
<tr>
<td>August</td>
<td>31-Jul-2013</td>
<td>2.9607</td>
</tr>
<tr>
<td>September</td>
<td>30-Aug-2013</td>
<td>2.9570</td>
</tr>
<tr>
<td>October</td>
<td>30-Sep-2013</td>
<td>2.9537</td>
</tr>
<tr>
<td>November</td>
<td>31-Oct-2013</td>
<td>2.9535</td>
</tr>
</tbody>
</table>

**TABLE 7:** Heating Oil Futures prices quoted on October 31, 2012 at ICE.

<table>
<thead>
<tr>
<th>3m</th>
<th>6m</th>
<th>9m</th>
<th>12m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fwd min</td>
<td>0.155</td>
<td>0.226</td>
<td>0.278</td>
</tr>
<tr>
<td>Fwd mean</td>
<td>0.157</td>
<td>0.228</td>
<td>0.281</td>
</tr>
<tr>
<td>Fwd max</td>
<td>0.159</td>
<td>0.232</td>
<td>0.285</td>
</tr>
<tr>
<td>Fwd curve</td>
<td>0.159</td>
<td>0.231</td>
<td>0.283</td>
</tr>
</tbody>
</table>

**TABLE 8:** Asian-stype option prices across maturities and varying assumptions about standing forward curve. (Jump frequency is assumed to be flat at level $\lambda_{\text{mean}}$.)

<table>
<thead>
<tr>
<th>3m</th>
<th>6m</th>
<th>9m</th>
<th>12m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_0$</td>
<td>0.129</td>
<td>0.186</td>
<td>0.228</td>
</tr>
<tr>
<td>$\lambda_{\text{min}}$</td>
<td>0.148</td>
<td>0.215</td>
<td>0.264</td>
</tr>
<tr>
<td>$\lambda_{\text{mean}}$</td>
<td>0.157</td>
<td>0.228</td>
<td>0.281</td>
</tr>
<tr>
<td>$\lambda_{\text{max}}$</td>
<td>0.165</td>
<td>0.241</td>
<td>0.297</td>
</tr>
<tr>
<td>$\lambda(t)$</td>
<td>0.160</td>
<td>0.234</td>
<td>0.288</td>
</tr>
</tbody>
</table>

**TABLE 9:** Asian prices across maturities and varying assumptions about jump frequency functions. Forward curve is assumed to be flat at the level “medium flat” defined as $\sum_{i=1}^{12} f_i \lambda_i = 2.9962$. 

energisk.org
- $\lambda_0 := 0$, which corresponds to continuous price paths;
- Time-varying intensity $\lambda(t)$.
- Lowest bound flat: $\lambda_{\min} := \min_{1 \leq i \leq 12} \lambda(i\Delta) = 3$.
- Medium level flat: $\lambda_{\text{mean}} := \frac{1}{12} \sum_{i=1}^{12} \lambda(i\Delta) = 4.5$.
- Greatest bound flat: $\lambda_{\max} := \max_{1 \leq i \leq 12} \lambda(i\Delta) = 6$.

For each of these assessments, we calculate option prices and plot the corresponding values against:

- Mean reversion frequency $\beta$: Figure 6 shows that option prices decrease with an increase in the speed at which prices tend to revert back to their long-term trend. In fact, higher mean reversion leads to smoothing jumps, a fact that reduces underlying price dispersions, so reducing the likelihood of ending up ITM.
- Brownian volatility $\sigma$: Figure 7 shows that option prices increase with an increase in the spot index volatility.
- Jump size mean $\xi$: Figure 8 shows that option prices increase with an increase in the jump size mean.
- Option moneyness defined, by convention, as $k/A_{0,i}$: Figure 9 shows that option prices decrease with an increase in the option moneyness.
- Time-to-maturity $T$: Figure 10 shows that option prices increase with an increase in the option lifetime.

Clearly, the greater the jump frequency, the higher the corresponding values for call options. Clearly, a zero jump intensity makes option prices independent of jump related parameters. This is the case reported with a red path in Figure 8.

We finally build four alternative assessment of the input forward curve:

- Market $\{F_{0,i\Delta}\}_{1 \leq i \leq 12}$, as we indicated earlier;
- Lowest flat: $\min_{1 \leq i \leq 12} F_{0,i\Delta} = 2.9535$;
- Medium flat: $\frac{1}{12} \sum_{i=1}^{12} F_{0,i\Delta} = 2.9962$;
- Greatest flat: $\max_{1 \leq i \leq 12} F_{0,i\Delta} = 3.0682$.

For each case, we report option prices against pairs of (maturity, input forward curve) and (maturity, input jump frequency function). Results are indicated in Tables 8 and 9, respectively.

Conclusions

We extend the semi-analytical price formula for Asian-style options derived in Fusai, Marena, and Roncoroni (2008) to the case of underlying spot prices driven by jump-diffusion processes. The key result is the calculation of the MGF for the spot price under these assumptions. Experiments conducted on market price data show that jumps may have a serious impact on the assessment of option prices despite the smoothing effect introduced by arithmetic averaging.

Future investigation might focus of the following spot price dynamics:

- Bivariate processes driven by a stochastic convenience yield;
- Multivariate processes with stochastic volatility;
- Jump-diffusions with random frequency of jumps.

Further extensions might encompass:

- Asian-style options written on a basket of prices;
- Convergence to continuous monitoring;
- Implied calibration on plain vanilla quotes.

ABOUT THE AUTHORS

Marina Marena Department of Economics and Statistics, Università degli Studi di Torino. Research interests are Mathematical Finance and Economic Dynamics.

Email address: marena@econ.unito.it

Andrea Roncoroni is full professor of Finance at ESSEC Business School, Paris, and Visiting Fellow at Bocconi University, Milan. Research interests are related to energy and commodity finance, risk monitoring and mitigation and applied quantitative modelling.

Email address: roncoroni@essec.edu

Gianluca Fusai Dipartimento di Studi per l’Economia e l’Impresa, Università del Piemonte Orientale, Novara (Italy), and Faculty of Finance, Cass Business School, London (UK). Research interests are Credit Risk and Counterparty Risk, Model Risk in Derivative Pricing, Electricity and Energy Markets.

Email address: fusai@eco.unipmn.it

The authors of the article would like to thank Energisk.org and QFinance for supporting this study, the Editor of Argo, and the referee for valuable comments. The usual disclaimers apply.

ABOUT THE ARTICLE

Submitted: March 2013.
Accepted: April 2013.
References


A General Approach to Modelling and Pricing in Energy and Weather Markets

The approach presented in this article results from some considerations about the stationary component of energy price models. General Ornstein-Uhlenbeck process is extended by more powerful stationary stochastic processes in order to match a wide range of different correlation effects.

Fred Espen BENTH

The article is divided into three main sections. The first introduces the basic theory of modeling with the comparison between mean-reverting and stationary stochastic processes: the Ornstein-Uhlenbeck process is suitable in explaining the dynamics of commodity prices but it presents some limits when a richer autocorrelation structure is required (for example in power price dynamics). The second section deepens in the stationary stochastic models indeed introducing Lévy processes able to extend the O-U process of the previous section. The last part of the article simply shows the theory applied to price derivative contracts written on energy commodities and weather markets.

Basics of energy and weather markets modelling

Mean reversion is a much used property in connection with commodity price modelling, prescribing that prices tend to a long term mean. A simple mean reverting stochastic dynamics is the classical Schwartz model (see Schwartz [7]) defined by

\[
\ln S(t) = \Lambda(t) + X(t)
\]

where \( S(t) \) is the price of some commodity at time \( t \), \( \Lambda(t) \) is some seasonal mean (that may be stochastic) and \( X(t) \) is a so-called Ornstein-Uhlenbeck process

\[
dX(t) = -aX(t)\,dt + \sigma\,dB(t)
\]

The process \( B \) is a Brownian motion, while \( a, \sigma \) are two positive constants, signifying the speed of mean reversion and the volatility, respectively.

If we for a moment let \( \sigma = 0 \), then we have the deterministic dynamics

\[
dX(t) = -aX(t)\,dt
\]

from which we see that \( X \) has a speed being proportional to \( X \) at time \( t \). The proportionality factor is \( -a \), and we see that the speed is positive if \( X(t) < 0 \), and negative when \( X(t) > 0 \). Hence, if the current value of \( X \) is bigger than zero, the process will have a push down towards zero, and opposite when \( X \) is below zero. Hence, the process reverts to the level zero, at a speed which increases the farther away we are from this level. This explains the label speed of mean reversion assigned to \( a \), and also why \( X \) is referred to as a mean reverting process.

Adding a noise \( \sigma\,dB(t) \) will introduce random “kicks” to the deterministic pathwise behaviour of \( X \). That is, if \( X > 0 \), the drift term will force \( X \) down towards zero, while \( \sigma\,dB(t) \) may either help in the reversion or give a push in the opposite direction. In fact, the process may move away from the mean level zero due to these random kicks.

Using an Ornstein-Uhlenbeck dynamics \( X \) in the modelling of the price dynamics of a commodity as in (1) says that the prices (on log-scale) will revert towards a mean level \( \Lambda(t) \), with an expected speed being proportional to the current state of \( X \).

---

20The work presented in this article has been financially supported by the Norwegian Research Council under the projects “Electricity Markets: Modelling, Optimization and Simulation” (EMMOS) and “Managing Weather Risk in Energy Markets” (MAWREM). The support is gratefully acknowledged.
There are reasons to question such an explicit physical behaviour of the prices, as the actual property we would like to model is stationarity. A stationary stochastic process will have paths concentrated around a mean, and as such is more flexible than a mean reverting process yet having a similar property of "a push towards a mean".

Stationarity is detected in for example to time evolution of weather variables and in power prices. Looking at a time series of temperatures observed in some location, one will find that they are stationary around a seasonal mean. The same goes for wind speeds, and we refer to [3] for detailed statistical analysis of such data. Indeed, data studies show that wind speeds and temperatures are well-described by higher-order autoregressive moving average (ARMA) processes, being stationary. Hence, an Ornstein-Uhlenbeck process, being the continuous-time version of a first-order autoregressive model, is too simplistic. [5] applies continuous-time ARMA models for temperature and wind speed as the basic models in pricing weather derivatives like futures and European call and put options. Futures contracts on temperature are typically settled with respect to a temperature index measured in a location. Such indices may be the cumulative evolution of weather variables and in power prices. Indeed, data studies show that wind speeds and temperatures are rich in structure for the autocorrelation.

Other energy prices like gas, oil or coal may include a non-stationary stochastic factor. This may be explained by limited resources available. We then let \( \Lambda(t) \) be some non-stationary stochastic process, for example a drifted Brownian motion. Our considerations here are focusing on the stationary part of such energy price models.

Stationary stochastic models

In [1] a general class of (semi-)stationary models extending the Ornstein-Uhlenbeck process in (2) is introduced, called Lévy stationary (LS) processes,

\[
Y(t) = \int_{-\infty}^{t} g(t-s) dL(s)
\]

Here, \( L \) is a Lévy process, which is a random walk in continuous time where the changes \( dL(t) = L(t + dt) - L(t) \) is not necessarily normally distributed. Further, \( g(x) \) is a positive deterministic function. If \( L = B \), then we recover \( X(t) \) in (2) by defining \( g(x) = \exp(-\alpha x) \). It holds that \( Y \) is a stationary process, and the correlation between \( Y(t) \) and \( Y(s) \) is proportional to

\[
\int_{0}^{\infty} g(|t-s| + v) g(v) dv.
\]

Thus, by an appropriate specification of \( g \) we can match a wide range of different correlation structures observed in the market. As the stationary distribution of \( Y \) can be characterised via the characteristics of \( L \), we can estimate the process \( y \) to data.

A particularly interesting and relevant special case of LS processes is the continuous-time autoregressive moving average (CARMA) processes. In the context of LS processes, they are defined by selecting

\[
g(x) = b' \exp(Ax) e_p
\]

where \( e_p \) is the \( p \)-dimensional unit vector with zeros everywhere except in the last coordinate, \( b \) is a \( p \)-dimensional vector with non-zero elements in the
first $q + 1$ coordinates and $A$ is the $p \times p$ matrix

$$A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
& \vdots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & \cdots & -\alpha_1
\end{bmatrix}$$

where $\alpha_i$ are positive constants. In the definition, we need to have $0 \leq q < p$. For given $(p,q)$, we call the process $Y$ with $g$ as above as a CARMA($p,q$)-process. In [5] it is shown how to explicitly link $Y$ to an autoregressive time series of order $p$ in the case $b = e_1$, the unit vector in $\mathbb{R}^p$ with 1 in the first coordinate and zeros elsewhere. The CARMA($p,q$)-processes share many of the properties of their discrete-time analogues, and is applied to temperature and wind speed modelling in [5] (using respectively, CAR(3,0) and CAR(4,0) processes) and to power spot prices in [3] and [1] (using CARMA(2,1)).

One may generalize the LS processes to allow for stochastic volatility as well. Such processes take the form

$$Y(t) = \int_{-\infty}^{t} g(t-s) \sigma(s) dL(s)$$

and is known as Lévy semistationary (LSS) processes. Here, $\sigma(s)$ is a stochastic process, and an efficient modeling class is obtained if one let $\sigma^2(t)$ be defined as a LS process driven by an class of positive Lévy processes.

One may ask how to simulate the paths of an LS process iteratively $Y$. An efficient algorithm can be developed from the integral representation

$$Y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}_\lambda(y) Y_{\lambda,y}(t) dy$$

where $\tilde{g}_\lambda$ is the Fourier transform of $g_\lambda(s) = \exp(\lambda x) g(x)$ for a positive constant $\lambda$ and $Y_{\lambda,y}(t)$ is a complex-valued Ornstein-Uhlenbeck process with speed of mean reversion $iy - \lambda$. Here, $i = \sqrt{-1}$, the complex unit. The paths of Ornstein-Uhlenbeck processes are fast to simulate iteratively using a standard Euler scheme, and by simulating such paths for different values of $y$, one can integrate numerically the expression above to obtain the path of $Y(t)$. Note that the simulation of $Y_{\lambda,y}(t)$ is not depending on $g$, and therefore one can do simulations for different choices of $g$ re-using the simulated paths of $Y_{\lambda,y}$. This is an advantage in estimation procedures (see [2] for details).

Pricing of forwards and derivatives

Consider a forward contract entered at time $t$ with delivery of some commodity at time $T \geq t$, and let the (log-)commodity price be defined as the sum of a seasonal function and an LS process $Y$. Then the forward price $f(t, T)$ is given explicitly as

$$f(t, T) = h(t, T, \theta) \exp \left( \int_{-\infty}^{t} g(T - s) dL(s) \right)$$

where

$$h(t, T, \theta) = \exp \left( \Lambda(T) + \int_{0}^{T-t} \phi_L(g(s) + \theta) - \phi_L(\theta) ds \right)$$

with $\phi_L(x)$ being the log-moment generating function of $L$. Here, $\theta$ is a constant called the market price of risk. It is directly linked to the risk premium of the contract, and may also be viewed as a function of the of the convenience yield in markets where this is relevant. In the power market, the convenience yield does not makes sense, as is the case of weather markets, since one cannot store the underlying (power or weather). In power markets, the forward is settled on the average of the spot price of electricity over a settlement period. Typically, this must be computed numerically. For temperature and wind forwards, there exist indices linked to the average or the aggregated underlying, which we also can compute as a (weighted) integral of $f(t, T)$ over $T$. Noteworthy, the forward price in (5) is not depending on the current state of $Y$, as we see that the integral in the exponent has an integrand $g(T-s)$, and not $g(t-s)$ as has to be the case if the forward price is a function of the current state $Y(t)$, or, in other words, the current state of the deseasonalized log-commodity spot price. However, if $Y$ is a CARMA($p,0$)-process, it turns out that the forward price can be expressed as a sum of the derivatives of $Y$ as follows (see [6] for details);

$$\ln f(t, T) = \ln h(t, T, \theta) + \sum_{i=0}^{p-1} f_i(T-t) Y^{(i)}(t)$$

with $Y^{(i)}(t)$ is the $i$th derivative of $Y$ at time $t$ and $f_i(x)$ are shape functions in time to maturity $x$ given by

$$f_i(x) = e_i \exp(A x) e_{i+1}$$

The forward price becomes a function of the deseasonalized log-spot and its derivatives, which means that the forward is dependent on the pathwise properties of the spot, and not only on the state itself. This is a result of incompleteness of the weather and power market in the sense that one cannot replicate the forward exactly by the buy-and-hold strategy. In Figure 1 we have shown the three shape functions in the case of $p = 3$ where the parameters of the $A$ matrix are collected from an
empirical fitting to temperature data (see [6]). For example, the (deseasonalized log-) spot price will be “transported” along the forward curve according to the shape function $f_0(x)$, which gives a decaying curve if $Y$ is positive. The derivative of $Y(t)$ will scale the shape $f_1(x)$, and this will give a hump upwards in the forward curve if the spot is on an upward trend (has a positive derivative). One can also price options on the forward price as given in (5). As one can compute the characteristic function of $Y$ and more generally of

$$\int_{-\infty}^{t} g(T-s) dL(s)$$

one can appeal to Fourier pricing techniques to obtain numerically efficient expressions for the price (see [1]).

ABOUT THE AUTHOR
Fred Espen Benth is professor of mathematical finance at the University of Oslo. Research interests are, among others, electricity, energy, weather markets, pricing and hedging of derivatives and portfolio optimization.

Email address: fredb@math.uio.no

ABOUT THE ARTICLE
Submitted: April 2013.
Accepted: May 2013.

REFERENCES


Trading Oil Spreads: Statistical Arbitrage

The article investigates statistical arbitrage trading opportunities in oil spreads, providing two key contributions. The first describes the application of an innovative statistical arbitrage trading model to the trading of oil spreads. The second instead is related to the implementation of recent procedures to control for the multiple comparisons problem in evaluating the performance of the statistical arbitrage trading strategies.

Mark CUMMINS
Andrea BUCCA

This article provides a brief synopsis of a recent paper of ours (Cummins and Bucca, 2012) that investigates statistical arbitrage trading opportunities in oil spreads, spanning a range of calendar, crack and locational spreads. The paper makes two key contributions in (i) the application of the novel optimal statistical arbitrage trading model of Bertram (2010) to the trading of oil spreads and (ii) the application of recently developed generalised multiple hypothesis testing procedures to control for the multiple comparisons problem (commonly referred to as data snooping bias in empirical finance applications) in evaluating the performance of the statistical arbitrage trading strategies.

Quantitative trading opportunities in the crude oil and refined products markets are investigated over the period 2003-2010, with particular focus on WTI, Brent, heating oil and gasoil. A wide range of common- and cross-commodity spreads are considered. Trading strategies are designed so as to exploit any predictability that exists in the unit volumetric spreads. The novel optimal statistical arbitrage trading model of Bertram (2010) is applied for the empirical analysis. Few papers deal directly with the issue of optimal entry and exit trading signals for statistical arbitrage trading strategies. These include Vidyamurthy (2004), Elliott et al (2005) and Do et al (2006). The approach of Bertram (2010) is very much distinct from these papers. Specifically, modelling a given spread series as a mean-reverting Ornstein-Uhlenbeck process, analytic solutions are shown to exist for the optimal entry and exit levels determined through maximising the expected return per unit time, which is defined as the ratio of the deterministic return of the strategy to the expected trade cycle time. Statistical arbitrage trading strategies with defined entry and exit levels offer deterministic returns but uncertainty lies in the length of the trade cycles. Normalisation by the expected trade cycle time explicitly accounts for the alternative deterministic returns and stochastic trade cycle times associated with alternative strategies, allowing for consistent cross-comparison.

The full set of commodity data considered in this study leads to 861 spreads in total. The objective is then to statistically test the performance of just over 2,500 statistical arbitrage style strategies simultaneously, based off the constructed spreads and using historical periods of one, two and three years for estimating the trading strategies. The hypothesis tests are designed to formally identify those trading strategies that, with statistical significance, outperform a given benchmark in terms of mean daily log-return. The benchmark is defined to have zero mean daily log-return, corresponding to taking no position in a given spread. This introduces the well-known issue of data snooping bias, whereby under naive analysis profitable trading strategies may be identified by pure chance alone. This links directly to the broader issue of multiple hypothesis testing in general statistical and econometric applications. Romano, Shaikh and Wolf (2009) provide a detailed exposition of the issues pertaining to multiple hypothesis test-
ing, outlining the key literature in the area. Two recent generalised stepwise techniques proposed in the literature are used to control for data snooping bias. The stepdown procedure of Romano and Wolf (2007) and the balanced stepdown procedure of Romano and Wolf (2010) are applied, both serving as improvements over more conservative single step approaches, such as the seminal reality check bootstrap test of White (2000) and the superior predictive ability test of Hansen (2005). The generalised procedures offer greater power, where power is loosely defined as the ability to reject false null hypotheses. The balanced stepdown procedure offers a further improvement in allowing for balance amongst the hypothesis tests in the sense that each is treated equally in terms of power. For further details of these methodologies, the interested reader is directed to the original papers of Romano and Wolf (2007) and Romano and Wolf (2010) or to Cummins and Bucca (2012).

A comprehensive data set of crude oil and refined product futures contracts is used, comprising WTI and Brent on the crude oil side and gasoil (GO) and heating oil (HO) on the refined products side. These commodities are chosen for this study on the basis of size, importance and liquidity. The data set covers the 11-year period from 3rd January 2000 to 31st December 2010. Most notably, this period set covers the 11-year period from 3rd January 2000 to 31st December 2010. Most notably, this period covers the record high crude oil prices recorded in 2008 and the subsequent collapse in the latter part of the same year resulting from the global economic crisis, in addition to the gradual recovery in crude oil prices over 2009-2010. All relevant conversions were done to ensure the time series are quoted consistently in dollars per barrel.

Optimal Statistical Arbitrage

This section provides a detailed mathematical exposition of the novel optimal statistical arbitrage trading model of Bertram (2010). The issue of optimal statistical arbitrage trading is approached by first assuming that the spread between two asset log-price series, denoted \( s_t \), is given by the following zero-mean OU process:

\[
d s_t = -\alpha s_t dt + \sigma dW_t,
\]

with \( \alpha, \sigma > 0 \) and \( W_t \) denoting a Wiener process. Defining the entry and exit levels of the trading strategy by \( a \) and \( m \) respectively, a complete trade cycle is the time taken for the spread process to transition from \( a \) to \( m \) and then return back to \( a \). Formally, the trade cycle time is defined as follows:

\[
T \equiv T_{a \rightarrow m} + T_{m \rightarrow a},
\]

where \( T_{a \rightarrow m} \) is the time to transition from \( a \) to \( m \) and \( T_{m \rightarrow a} \) is the time to transition from \( m \) to \( a \), and the independence of the two times follows from the Markovian property of the OU process. Given relative transaction costs \( c \), the total log-return from one trade cycle of the statistical arbitrage trading strategy is given by \( r(a, m, c) \equiv m - a - c \), which is deterministic but for which the associated trading cycle time is stochastic. In this context, Bertram (2010) proposes the following expected return per unit time and variance of return per unit time measures:

\[
\xi (a, m, c) \equiv \frac{r(a, m, c)}{E(T)}, \\
\varsigma (a, m, c) \equiv \frac{r^2(a, m, c) V(T)}{E^3(T)}
\]

where \( E(T) = E(T_{a \rightarrow m}) + E(T_{m \rightarrow a}) \) is the expected trade cycle time and \( V(T) = V(T_{a \rightarrow m}) + V(T_{m \rightarrow a}) \) is the variance of the trade cycle time. Following a transformation of the OU process to a dimensionless system, and drawing on first-passage time theory, Bertram (2010) derives analytic expressions for \( E(T) \), \( V(T) \), \( \xi (a, m, c) \) and \( \varsigma (a, m, c) \). Further details of these analytic solutions may be found in the original paper of Bertram (2010) or in Cummins and Bucca (2012).

With these analytic results in place, it is shown that the optimal entry and exit levels \( a^* \) and \( m^* \) may be derived by either maximising the expected return per unit time \( \xi (a, m, c) \) or the associated per unit time Sharpe ratio. In the former case, it is established that \( m^* = -a^* \), with \( a^* < 0 \) being the root of the equation

\[
\exp \left( \frac{aa^2}{\sigma^2} \right) (2a + c) - \sigma \sqrt{\frac{\pi}{a}} \text{Erfi} \left( \frac{a\sqrt{a}}{\sigma} \right) = 0.
\]

In the latter case, the risk-free rate of interest \( r_f \) is introduced and the associated per unit time Sharpe ratio is given by

\[
S(a, m, c, r_f) \equiv \frac{\xi (a, m, c) - r_f}{\varsigma (a, m, c)}
\]

\[
= \left( m - a - c - r_f \right) \sqrt{\frac{E(T)}{(m - a - c)^2 V(T)}}.
\]

It is established similarly that \( m = -a, a < 0 \), and the optimal entry level \( a^* \) follows from maximising the Sharpe ratio expression with this substitution for \( m \).
Multiple Hypothesis Testing

The objective of the study is to formally and statistically test the performance of the optimal statistical arbitrage trading model of Bertram (2010) described in the previous section for the quantitative trading of spreads in the crude oil and refined products markets. This will inevitably involve the testing of a large number of implementations of the trading model simultaneously. This introduces the well-established issue of data snooping bias, whereby under naive analysis profitable trading strategies may be identified by pure chance alone. This links directly to the broader issue of multiple hypothesis testing in statistical and econometric applications. The remainder of the section expands on this, and introduces two key generalised techniques that will be used to control for the problem of data snooping bias.

The issue with multiple hypothesis testing is that the probability of false discoveries, i.e. the rejection of true null hypotheses by chance alone, is often significant. There are a number of approaches described in the literature to deal with this multiple comparisons problem and control for the familywise error rate and variants of this. Romano et al (2009) provide an excellent summary of the issues and the literature. The familywise error rate (FWER) is defined as the probability that at least one or more false discoveries occur. Consistent with the notation of Romano et al (2009), the following definition is made: \( FWER_\theta = P_\theta \{ \text{one or more false discoveries} \} \). Controlling the FWE involves setting a significance level \( \alpha \) and requiring that \( FWER_\theta \leq \alpha \). Controlling for the FWE in this way is particularly conservative given that it does not allow even for one false discovery and so is criticised for lacking power, where power is loosely defined as the ability to reject false null hypotheses, i.e. identify true discoveries (Romano et al, 2009). The greater \( S \), the more difficult it is to make true discoveries.

To deal with this weakness, generalised FWE approaches have been proposed in the literature. The generalised FWE seeks to control for \( k \) (where \( k \geq 1 \)) or more false discoveries and, in so doing, allows for greater power in multiple hypothesis testing. The generalised \( k \)-FWE is defined as follows: \( k-FWER_\theta = P_\theta \{ k \text{ or more false discoveries} \} \).

Towards building a framework to identify profitable trading strategies, with statistical significance, on the set of calendar and crack spreads in this study, the following one-sided hypothesis tests will be considered:

\[ H_{0,\delta} : \theta_\delta \leq 0 \quad \text{vs.} \quad H_{1,\delta} : \theta_\delta > 0. \]

The objective is to control for the multiple comparisons in this scenario through the generalised familywise error rate, which offers greater power while also implicitly accounting for the dependence structure that exists between the tests. On this basis, the stepwise procedure of Romano and Wolf (2007) is implemented, which serves as an improvement over single step approaches by allowing for subsequent iterative steps to identify additional hypothesis rejections. A further improvement is implemented in the form of the balanced stepwise procedure of Romano and Wolf (2010). This procedure is a marked improvement again that allows for balance amongst the hypothesis tests in the sense that each is treated equally in terms of power, i.e. in the identification of true discoveries. So in general terms, trading strategies with “large” profitability are not allowed to dominate those trading strategies with “small” profitability in the testing procedure.

Empirical Results

For full details of the empirical implementation, please refer to Cummins and Bucca (2012). The empirical results obtained lead to a number of interesting observations. Tables 1 and 2 present trading performance results based on applying the stepdown and balanced stepdown procedures respectively to control for data snooping bias. For each trading year, average daily returns and Sharpe ratios are reported for three specific categories; namely, the top ten, top twenty and all trading strategies identified by the procedures as being profitable with statistical significance. The third column in each table gives the total number of profitable trading strategies identified in each year.

It is important to emphasise that the results reported reflect the aggregation of two trading approaches, one taking long positions in the spreads to exploit upward movements between the trading signals and the second taking short positions to exploit downward movements between the trading signals. Many profitable trading strategies are identified, which in many instances report Sharpe ratios that exceed 2 and in some instances are even close to 4. For the top ten and top 20 categories, average daily returns fall within the approximate range of 0.07%-0.35% for most years, with 2009 showing exceptional results in the range of 0.5%-0.55% driven by the locational HO-GO spreads. The associated trade lengths lie in the approximate range of 9-55 trade dates, with the shortest horizons being reported for 2009. The years 2003 and 2009 prove to be particularly successful relative to other years with Sharpe ratios close to or in excess of 3. For the years 2008 and 2010, the stepdown procedure
### TABLE 1: Average Empirical Results: Stepdown Procedure

<table>
<thead>
<tr>
<th>Year</th>
<th>Avg. Daily Ret.</th>
<th>Avg. Sharpe Ratio</th>
<th># Profitable Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>2003</td>
<td>Top 10 : 0.00361</td>
<td>2.94</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00345</td>
<td>2.92</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00345</td>
<td>2.92</td>
<td>15</td>
</tr>
<tr>
<td>2004</td>
<td>Top 10 : 0.00299</td>
<td>2.44</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00275</td>
<td>2.30</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00265</td>
<td>2.24</td>
<td>28</td>
</tr>
<tr>
<td>2005</td>
<td>Top 10 : 0.00259</td>
<td>2.33</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00253</td>
<td>2.22</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00253</td>
<td>2.22</td>
<td>14</td>
</tr>
<tr>
<td>2006</td>
<td>Top 10 : 0.00241</td>
<td>2.41</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00220</td>
<td>2.19</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00195</td>
<td>1.98</td>
<td>35</td>
</tr>
<tr>
<td>2007</td>
<td>Top 10 : 0.00293</td>
<td>2.93</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00272</td>
<td>2.75</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00233</td>
<td>2.32</td>
<td>54</td>
</tr>
<tr>
<td>2008</td>
<td>Top 10 : -</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : -</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : -</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2009</td>
<td>Top 10 : 0.00547</td>
<td>3.95</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00499</td>
<td>3.66</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00456</td>
<td>3.32</td>
<td>31</td>
</tr>
<tr>
<td>2010</td>
<td>Top 10 : -</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : -</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : -</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

### TABLE 2: Average Empirical Results: Balanced Procedure

<table>
<thead>
<tr>
<th>Year</th>
<th>Avg. Daily Ret.</th>
<th>Avg. Sharpe Ratio</th>
<th># Profitable Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>2003</td>
<td>Top 10 : 0.00354</td>
<td>2.96</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00314</td>
<td>2.87</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00164</td>
<td>2.16</td>
<td>142</td>
</tr>
<tr>
<td>2004</td>
<td>Top 10 : 0.00298</td>
<td>2.47</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00272</td>
<td>2.34</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00174</td>
<td>1.84</td>
<td>52</td>
</tr>
<tr>
<td>2005</td>
<td>Top 10 : 0.00258</td>
<td>2.33</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00246</td>
<td>2.22</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00202</td>
<td>2.05</td>
<td>71</td>
</tr>
<tr>
<td>2006</td>
<td>Top 10 : 0.00241</td>
<td>2.41</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00220</td>
<td>2.19</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00164</td>
<td>2.03</td>
<td>46</td>
</tr>
<tr>
<td>2007</td>
<td>Top 10 : 0.00293</td>
<td>2.93</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00272</td>
<td>2.75</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00199</td>
<td>2.18</td>
<td>82</td>
</tr>
<tr>
<td>2008</td>
<td>Top 10 : 0.00244</td>
<td>1.86</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00244</td>
<td>1.86</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00244</td>
<td>1.86</td>
<td>8</td>
</tr>
<tr>
<td>2009</td>
<td>Top 10 : 0.00547</td>
<td>3.95</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00499</td>
<td>3.66</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00264</td>
<td>2.43</td>
<td>63</td>
</tr>
<tr>
<td>2010</td>
<td>Top 10 : 0.00092</td>
<td>1.91</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Top 20 : 0.00066</td>
<td>1.82</td>
<td></td>
</tr>
<tr>
<td></td>
<td>All : 0.00060</td>
<td>1.73</td>
<td>23</td>
</tr>
</tbody>
</table>
actually fails to identify any profitable strategies, with the balanced stepdown procedure only extracting 8 and 23 profitable strategies respectively. The numbers are much lower than in other years. Average daily returns in excess of 0.2% are reported for 2008 but the associated Sharpe ratios are seen to fall below 2 on average. The discussion to follow will explore further these observations for 2008 in the context of the differences between the stepdown and balanced stepdown procedures. Similar poor performance is observed for the year 2010, where average daily returns can be seen to be at the lower range of approximately 0.07%, with Sharpe ratios again below 2 on average.

Substantial differences are observed between the stepdown and balanced stepdown procedures in terms of the total number of profitable trading strategies identified, despite showing broad consistency within the top 10 and top 20 categories. The balanced stepdown procedure manages to identify between 1.5 and 9.5 times the number of profitable strategies and succeeds in identifying strategies in 2008 and 2010 where the stepdown procedure fails. This observation reflects the manner in which the balanced stepdown procedure treats each hypothesis test equally in terms of power, whereas the stepdown procedure is biased towards those trading strategies with larger average daily returns. The dramatic reduction in the number of profitable strategies in 2008 relative to 2007 and the years previous reflects the collapse in crude oil and other commodity prices resulting from the credit crisis shock and concerns over global commodity demand. The effect of the collapse in commodity prices manifests as a structural shift in the range of the spreads over 2008 relative to 2007, 2006 and 2005. Therefore, large divergences from the long-run mean levels estimated in-sample are observed for the majority of spreads. These divergences lead to significant losses for the majority of trading strategies out-of-sample in 2008. For more granular insights into the performance of the trading strategies, the interested reader is directed to the full paper of Cummins and Bucca (2012).

ABOUT THE AUTHORS
Mark Cummins is a Lecturer in Finance at the Dublin City University Business School and Programme Chair of the MSc in Sustainable Energy Finance. Research interests are numerical methods in Finance, energy modeling, risk management and trading.
Email address: mark.cummins@dcu.ie
Andrea Bucca is working at Glencore Intl Plc. Research interests are energy and commodity finance, econometrics and shipping freight.
Email address: andrea.bucca@glencore.co.uk

ABOUT THE ARTICLE
Submitted: May 2013.
Accepted: June 2013.

REFERENCES
Crash Course

Monetary Measurement of Risk

Part I: General definitions and VaR
Monetary Measurement of Risk: a Critical Overview

Part I: General Definitions and Value-at-Risk

This is the first of a series of articles that will interest practitioners of both finance and energy industry: Lionel Lecesne and Andrea Roncoroni provide a critical overview of the monetary measurement of risk. In particular in Part I they introduce general definitions of risk assessment and risk measures, then focusing on the most famous monetary measure of risk: Value-at-Risk.

Lionel LECESNE
Andrea RONCORONI

The simplest and perhaps most significant definition of risk relates the term to any sort of “exposure to uncertainty” (Leppard (2005)). We hereby focus on risk affecting physical and financial asset values and related cash flows, what we refer to as financial risk. Our starting point is the notion of position intended as a set, possibly a singleton, of assets generating financial cash flows as a primary or secondary output. Assets of this kind include financial securities, loans, commodity portfolios, commercial agreements, and real assets. Our goal is to develop a critical introduction to the class of monetary measures of risk, i.e., cash-valued metrics of risk affecting a position under consideration.

Risk Assessment

Risk assessment may be thought of as a conceptual part of any risk management process. More precisely, it is a key step following risk identification (and qualification) and preceding risk mitigation.

RISK MANAGEMENT PROCESS

Identification → Measurement → Mitigation

To give a feeling of the kind of “object” we are concerned with, let us begin by introducing major qualitative classes of risk affecting financial assets. We may consider five major classes of risk:

A. Market risk concerns exposure of asset value (and related cash flows) to market variables. These variables include quoted figures and over-the-counter prices. As an example, we may consider an investor who buys stock shares for 100 Euros; due to upcoming bad news concerning the underlying company, the stock price might drop down to 90 Euros on the following week; as a consequence, the marked-to-market (MTM) value of the investor’s position would possibly drop down by 10 Euros, a loss he would experience provided that he decides (and manages) to close his position out through stock selling at the new market price.

B. Credit risk concerns exposure of asset value (and related cash flows) to credit-linked events. These events primarily comprise counterpart default events such as payment delays or unilateral changes of settlement conditions. As an example, we may consider a bank lending 30,000 Euros within a consumption credit deal; the client might lose his job and then he stop complying with the loan repayment schedule, a fact triggering a loss for the lending institution in the amount of the outstanding value of the underlying loan.

C. Liquidity risk refers to potential discrepancies between standing market quotes (or MTM values, depending on whether the position is
publicly quoted or not) and those at which actual trade take place. As an example, we may consider an investor wishing to quickly sell out his portfolio consisting of five identical bonds; a single bond quotes at 1,000 Euros, whereas a price of 1,000 Euros is bid for two bonds and a quote of 980 Euros is bid for three additional bond units; the investor would then get 2 × 1000 + 3 × 980 = 4940 Euros as opposed to the MTM portfolio value of 5 × 1000 = 5000 Euros. Liquidity risk may be particularly remarkable for commitments engaging parties to physical delivery of energy sources, raw materials, and other commodities.

D. **Operational risk** refers to exposure of asset values (and related cash flows) to “inadequate or failed internal processes, people and systems, or from external events” (Embrechts, McNeil, and Frey (2005)). For instance, a computer bug may lead a trader to assume an erroneous position leading to unexpectedly large losses. One of the most relevant source of operational risk affecting a financial position is represented by the procedures followed to operate quantitative models.

E. **Model risk** refers to potential discrepancies between the (unknown) actual value of an asset and fair values computed using varying models specifications. Theses specifications may derive either from alternative model structures or from alternative estimation methods.


**Risk Measures**

Once risk has been identified (and qualified), then we have to measure, *i.e.*, quantify its extent. Quantifying risk should be done in a way that allows for:

1. Comparing among assets in terms of the risk they borne, and:

   2. Assessing the absolute quality of an asset in terms of risk severity.

Broadly speaking, a **measure of risk** (or **risk metrics**) \( \rho \) is a function whose value quantifies the risk (of a certain type) borne by the position under exam. There is no general consensus on the precise definition of risk metrics. Three are the main elements we must specify upon constructing a risk measure:

- **Domain** may be an outstanding position (*i.e.*, portfolio composition), its fair value, or any other quantity that is representative of that position (*e.g.*, cumulated cash flow). In the vast majority of cases, risk metrics applies to the future fair value of the position, possibly centered to the current MTM or fair value (*i.e.*, the position’s P&L).

- **Codomain** usually is a positive number whose meaning may vary according to the instance. The most popular measure of risk used in financial markets is **price volatility**. This quantity measures the average deviation of price variations (or return) from the corresponding average value. We instead focus on a class of risk measures allowing to express risk in terms of monetary value. This approach is particularly intuitive and convenient for two reasons, at least: first, risk assessment becomes tightly linked to the traditional notion of **reserves**, namely capital allocated to face negative outcomes stemming from exposure to uncertainty; second, it provides the user with an immediate interpretation of risk extent in term of cash amount.

- **Properties** define rules that risk measures must comply with. These properties are based on **metaproperties** derived either from intuition or from goals we may wish to achieve using the measure.

We would like to set up a reasonable definition of **monetary measure of risk** \( \rho^{mon} \) over the set \( X \) of random variables representing positions’ P&L’s. We therefore assume that any position is represented by the corresponding P&L (and the two terms can be used interchangeably).

The attribute “monetary” refers to the interpretation of \( \rho^{mon} (X) \) as a cash amount, or reserves level, insuring the holder against the risk borne by position \( X \). This interpretation suggests that two positions exhibiting strictly ranked P&L’s actually require reserves following the opposite order relationship. This leads to a first requirement for monetary measures of risk, a property we refer to as:

- **Anti-monotonicity** (AM) : for any pair of positions \( X, Y \in X \), with \( X \leq Y \):

\[
\rho^{mon} (X) \geq \rho^{mon} (Y).
\]

Another consequence of the above mentioned interpretation is that adding a fixed cash amount \( m \) to a
position $X$ ought to allow reducing the corresponding reserves by the same amount. This leads to a second requirement for monetary measures of risk, a property we refer to as:

- **TRANSLATIONAL ANTI-VARIANCE** (TA): for any pay-off $X \in \mathcal{X}$ and cash amount $m > 0$:
  $$\rho_{\text{mon}}(X + m) = \rho_{\text{mon}}(X) - m.$$  

**Definition.** A Monetary Measure of Risk is an anti-monotonic and translational anti-variant function $\rho$ transforming any position $X$ into cash amount $\rho(X)$.

Monetary measures of risk are very simple to interpret. They offer a reply to a key question for any risk bearing operator: "For a risky position $X$, how much money, say $m$, shall I have to additionally hold in order to offset the negative possible consequences stemming from that risk, i.e., $\rho_{\text{mon}}(X + m) = 0$?" Using property (TA), we easily see that the required cash amount exactly matches $\rho_{\text{mon}}(X)$. Hence, the value assumed by any monetary measure of risk can be interpreted as cash reserve. In the banking industry, this kind of quantities is connected to the notion of Economic Capital representing the amount of money that prevents the business from bankruptcy in a predefined worst-case scenario of market evolution. The related amount is also referred to as Regulatory Capital within the Basel II regulatory framework, a set of internationally recognized banking rules aiming at preventing economies from experiencing systemic crises.

**Value-at-Risk**

Value-at-Risk (VaR) is undoubtedly the most famous monetary measure of risk. It subsumes the concept of maximum loss that a financial position may experience over a certain period of time $T$ up to an assigned level of confidence $\alpha \in [0, 1]$.

Let a position under concern be represented by a random variable $X$ with probability distribution $F_X$: positive (resp. negative) values of $X$ denote capital gains (resp. losses) accrued over a selected time horizon $T$ in the future. Notice that horizon $T$ is implicitly referred to in our former definition of position’s P&L $X$, hence we omit its indication in the symbol representing VaR. The user is required to indicate a confidence level $\alpha$.

Our metadefinition of VaR is equivalent to saying that $\alpha$ matches the probability of P&L $X$ exceeding $-\text{VaR}$:

$$\alpha = \mathbb{P}[\text{"loss"} - X \leq \text{VaR}] = \mathbb{P}[\text{"P&L"} X \geq -\text{VaR}].$$

This observation allows us to equivalently define VaR either as the $\alpha$-quantile (i.e., inverse function) of the loss distribution $F_{-X}$ or as the $(1 - \alpha)$-quantile of the P&L distribution $F_X$. For the sake of simplicity, let us consider P&L’s with strictly monotonic continuous distribution functions.

**Definition 1.** Given a random position $X$ and a confidence level $\alpha \in [0, 1]$, the $100\alpha\%$ Value-at-Risk of $X$ is given by:

$$\text{VaR}_\alpha(X) := F_{\text{loss} - X}^{-1}(\alpha),$$

$$:= F_{\text{P&L}}^{-1}(1 - \alpha).$$

**Remark 1.** In the case of P&L’s with strictly monotonic continuous distribution functions we have:

$$\text{VaR}_\alpha(X) := \inf \{x \in \mathbb{R} : \mathbb{P}[\text{"loss"} - X \leq x] = \alpha, \text{ or } \mathbb{P}[\text{"P&L"} X \geq -x] = \alpha, \text{ or } \mathbb{P}[\text{"P&L"} X < -x] = 1 - \alpha \}.$$ 

For weakly monotonic continuous distributions, the right-hand side of this definition must be encapsulated into an “inf” operator over real numbers, i.e.:

$$\text{VaR}_\alpha(X) := \inf \{x \in \mathbb{R} : \mathbb{P}[\text{"loss"} - X \leq x] = \alpha \}.$$  

For general distributions (e.g., discontinuous functions), equality condition “$\equiv$” must be replaced by the weaker “$\geq$”:

$$\text{VaR}_\alpha(X) := \inf \{x \in \mathbb{R} : \mathbb{P}[\text{"loss"} - X \leq x] \geq \alpha \}.$$ 

These extensions correspond to the formula reported in the above definition provided that $F^{-1}$ represents appropriate generalized inverse functions guaranteeing existence and uniqueness of VaR.

**Proposition 1.** VaR is a monetary measure of risk.

**Proof.** (AM) Let $X > Y$. Due to increasing monotonicity of probability distribution functions, AM statement $\text{VaR}_\alpha(X) < \text{VaR}_\alpha(Y)$ is equivalent to claiming that $F_{-X}(\text{VaR}_\alpha(X)) < F_{-X}(\text{VaR}_\alpha(Y))$. This latter follows from:

$$\mathbb{P}[{-X \leq \text{VaR}_\alpha(X)}] \equiv \alpha \equiv \mathbb{P}[{-Y \leq \text{VaR}_\alpha(Y)}] < \mathbb{P}[{-X \leq \text{VaR}_\alpha(Y)}],$$

where the inequality stems from noting that $X > Y \Leftrightarrow -X < -Y \Leftrightarrow \{-Y \leq x\} \subset \{-X \leq x\}$ for any $x$ (in particular for $x := \text{VaR}_\alpha(Y)$).

---

21Here 0 stands for no confidence, 1 indicates full confidence, and any value $a$ comprised between the two is interpreted as being confident at 100%.

22Symbol “$\equiv$” stands for definition: “$a := b$” reads as 1) $b$ is a mathematically meaningful expression and 2) symbol “$a$” is set to represent $b$. Symbol “$\equiv$” stand for identity: “$a \equiv b$” reads as 1) $a = b \iff c = d$ and 2) $c := d$ for suitable $c$ and $d$.  

Winter 2014 69
Let \( m > 0 \) be a certain cash amount. TA statement \( \text{Var}_\alpha(X + m) = \text{Var}_\alpha(X) - m \) follow from unicity of VaR combined with the following equality:

\[
P[-(X + m) \leq \text{Var}_\alpha(X + m)] = \alpha = P[-X \leq \text{Var}_\alpha(X)] = P[-(X + m) \leq \text{Var}_\alpha(X) - m],
\]

where the last equality stems from subtracting \( m \) from both sides of the inequality within squared brackets.

Figure 1 exhibits VaR across values taken by standard deviation (parameter denoted as “sigma” and representing P&L volatility) of the underlying probability distribution, which we assume to be a standard normal, and confidence level (parameter denoted as “alpha”). The 3-dimensional graph clearly shows that VaR monotonically increases with the volatility for large values of confidence level. This property is in agreement with the intuition whereby an increase of P&L volatility is expected to entail a raise in the economic capital to cover against the corresponding higher risk level. The same picture shows that VaR is as well monotonically increasing with the confidence level for any value of sigma. Indeed, higher levels of confidence would intuitively require more important economic reserves.

We conclude this section by developing a critical analysis of a few common statements about the adoption of VaR as a risk metric.

1. **VaR is simple.** Indeed it is a single number, which is relatively easy to compute and interpret. This undoubtedly is a quality of VaR in terms of effectiveness.

2. **VaR is grounded on a pessimistic view of the future.** The fact that VaR measures the extent of potential losses does not prevent the user from contextually look after expected rewards.

3. **VaR is an oxymoron** in that it measures the extent of extreme potential losses using statistically estimated distributions, hence under normal market conditions. This statement is correct to the extent that VaR is calculated using distributions estimated to past data only; market practice usually computes VaR using distributions estimated from a combination of past data (=statistical estimation), current quotes (=implied calibration), and personal beliefs.

4. **VaR applies to the wrong portfolio.** Indeed VaR calculation usually assumes that the portfolio stays unchanged all throughout the time horizon, a fact that is rejected by ex post empirical observation. If VaR is used to compare alternative portfolios, then this criticism is rather mild. It would possibly hold true in cases where portfolio composition may have a direct connection to the ability to add new positions in the future. If VaR is used for capital requirements, it would undoubtedly be more significant to dispose of some sort of forecast for portfolio composition to come. However,
lack of this piece of information does not depend on the adopted monetary measure of risk, nor does it on VaR, in particular.

5. VaR increases with time horizon. As a counterexample to this statement, consider a contract for delivery of heating oil. It would be reasonable to let the volume increase during cold season and shrink during warm season. This way, the time increasing uncertainty affecting the commodity price trades-off with the underling season volume. It may well be that exposure for the forthcoming wintertime exceeds the one forecasted for the following summertime.

6. VaR does not account for risk diversification. This point will be dealt with in greater detail in the next section. Here, we just sketch the issue. The idea of diversification concretizes into the statement whereby the additive combination of two positions, say P&L A and B, amounts to no increase in the sum of the exposure assessments of the each position individually taken. More formally, VaR ought to be sub-additive:

$$VaR_\alpha (A + B) \leq VaR_\alpha (A) + VaR_\alpha (B).$$

This property can be disproved for VaR by using an example we develop in the Part II of this article (see the next issue of this review). Most of the existing literature on VaR considers lack of sub-additivity a pitfall of VaR as a measure of risk. In our view, there is no reason underpinning diversification upon additive composition. More importantly, violations of sub-additiveness would signal particular cases for which the user ought to take a particular care of. Also, liquidity constraints might result into super-additiveness of risk assessments. We may then reasonably conclude that lack of sub-additivity improves the risk assessment power of a given metric.

7. VaR might not match actual losses. It is clear that P&L distribution assessment is a process possibly subject to model and estimation errors. More importantly, a VaR loss occurs provided that the corresponding positions are instantaneously wound up. However, transaction costs, trading delays, or even inability to trade at the marked-to-market value (e.g., a physical asset such as a gas powered power plant) might hinder the corresponding operation. Hence, this critic seems to have a solid ground.

8. VaR is insensitive to (i.e., ignores) the severity of losses beyond the confidence level. Consider the case of two positions sharing the same VaR assessment at, say, 95% of confidence. Assuming that in the 5% of worst cases the two P&L’s produce distinct losses. Would you still consider these positions equivalent in terms of risk bearing? This simple case shows the appropriateness of the claim whenever VaR is used for comparison purposes.

9. VaR assessment hides the true problem of risk measurement. This point refers to the fact that VaR charges parameters $T$ (=horizon) and $\alpha$ (=confidence level) the burden of risk assessment. Despite the apparent clarity of these two numbers, there is no general rule, theory or consensus on a rational way to determine their value. Besides the case of regulatory compliance, whereby VaR parameters are assigned by law, risk managers solely dispose of a number of rules of thumb such as:

(a) Link time horizon $T$ to market liquidity. For instance, one may equal it to the “typical” time horizon in the market where the core business is conducted. In the case of an insurance company this might equal one calendar year. One should also prefer near time horizons in light of concerns related to model estimation and linear approximation of asset values. Basel committee fixes it equal to 10 days, while the trading practice recommend a 1-day time horizon.

(b) Confidence level is set to 99% by the Basel Committee for capital requirement in the Banking industry, while traders’ limits usually adopt a value of 95%.

We believe that the last issue above stated represents one of the most serious drawbacks of VaR. A possible way out has recently been put forward at Energy Risk Europe 2010 Conference (Cesa (2010)): Andrea Roncoroni and Gianluca Fusai proposed the adoption of a new notion of FloVaR$^{TM}$ which amplifies the extent of VaR in a way that does not require any prior specification about time horizon and confidence level.\footnote{See www.energisk.org}

ABOUT THE AUTHORS
Andrea Roncoroni is full professor of Finance at ESSEC Business School, Paris, and Visiting Fellow at Bocconi University, Milan. Research interests are related to energy and...
commodity finance, risk monitoring and mitigation and applied quantitative modelling.
Email address: roncoroni@essec.edu

Lionel Lecesne is Ph.D. candidate at University of Cergy, France. He was formerly student at Ecole Normale Supérieure in Cachan, France. Research interests include risk measurement, energy finance and financial econometrics.
Email address: lionel.lecesne@u-cergy.fr

ABOUT THE ARTICLE
Submitted: June 2013.
Accepted: July 2013.

References

GARP ERP® Certification Preparation Course
Examination preparation for Energy Risk Professional certification by the Global Association of Risk Professionals (GARP)

London - Wed 16th - Fri 18th October 2013
Geneva - Thu 24th - Sat 26th October 2013

Hosting Institutions
A COMPREHENSIVE COURSE ON “PHYSICAL AND FINANCIAL RISK MANAGEMENT” IN PREPARATION FOR THE GARP ERP® CERTIFICATION EXAM BY THE GLOBAL ASSOCIATION OF RISK PROFESSIONALS (GARP)

About the ERP® Program
Get Certified as an Energy Risk Professional (ERP®)

The Energy Risk Professional (ERP®) Program is the first and only designation for global energy professionals. Energisk’s Course on “Physical and Financial Risk Management” fully covers the material in accordance with the topics outlined in the 2013 ERP Study Guide, and reflects the learning objectives defined by the GARP’s Energy Oversight Committee (EOC). This course will be an excellent preparation for practitioners and professionals planning to take the Energy Risk Professional (ERP®) Exam, with the course being delivered by experts in the energy space. Statistics says that 84% of Certified ERP® would prefer to hire a candidate with the ERP® designation over one without the credential, creating a powerful, elite network.

*http://www.garp.org/erp/career-benefits.aspx

FAST FACTS
» Since its inception, ERP® registrations have grown 99% year-on-year
» In 2011, ERP® candidates originated from 36 countries and over 450 organizations
» 163 organizations have been represented by three or more ERP® candidates since 2009
» 94% of those who sat for the November 2011 ERP® Exam reported that the topics were applicable to their current jobs

LANGUAGE
Course is delivered in English

SCHEDULE
Each day 09.00-18.00, including breaks

REGISTRATION
Process: www.energisk.org

CONTACTS
E-mail: contact@energisk.org
Web: www.energisk.org

MULTIPLE REGISTRATION DISCOUNT
Register two or more people from the same company and receive an additional 10% discount.

FEES FOR PROFESSIONALS (*)
» Early Bird € 1559 (before 20 September)
» Standard Price € 1779
» Students registered to fulltime programs at hosting institutions € 559

DOCUMENTATION
Participants receive documentation in a self-contained and comprehensive manual.

PRATICAL INFORMATION

London
DATES
16-18 October 2013
LOCATION
Cass Business School
London, United Kingdom

ORGANIZING INSTITUTIONS

Geneva
DATES
24-26 October 2013
LOCATION
University of Geneva
Geneva, Switzerland

ORGANIZING INSTITUTIONS
Energisk.org, Paris, in collaboration with University of Geneva

(*) FEES FOR PRIVATE INDIVIDUALS: Special tariff (please contact us)

GARP Disclaimer: GARP does not endorse, promote, review or warrant the accuracy of the products or services offered by Energisk of GARP Exam related information, nor does it endorse any pass rates that may be claimed by the Exam Prep Provider. Further, GARP is not responsible for any fees or costs paid by the user to Energisk nor is GARP responsible for any fees or costs of any person or entity providing any services to Energisk. ERP®, FRM, GARP®, and Global Association of Risk Professionals, ICBR® and FRBR™ are trademarks owned by the Global Association of Risk Professionals, Inc.