Analytical Credit VaR with Stochastic Probabilities of Default and Recoveries

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Abstract

We extend the model presented in Bonollo *et al.* [3] by introducing a multi-scenario framework that allows for a richer and more realistic specification, including non-static (stochastic) probabilities of default and losses given default. Though more complex from a computational point of view, the model with scenarios is still tractable analytically, yielding results in closed form expressions. The approximated value at risk has been calculated by generalizing the procedure exposed in [3] for the single scenario case, in the presence of granularity in the exposures, sector concentration and contagion. The outcome is not simply a weighted sum of the VaRs in the individual scenarios, but results in a more involved function of the single scenarios' parameters.

The theoretical model description is complemented with an in-depth numerical analysis.

1 Introduction

In recent years many models have been designed in theory and utilized in practice to calculate the value at risk (VaR) of credit portfolios. Though starting from different assumptions on the probability distributions of the factors affecting the default's occurrence of the obligors and on the recovery fraction of the exposures at default, all these models try to measure also the concentration and contagion risks. Example of such models are CreditMetrics [16], CreditRisk+ [27], PortfolioManager [2] and CreditPortfolioView [29] and some of them rely on computationally heavy Monte Carlo simulations to work out the calculations.

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A different solution to the problem of calculating economic capital for credit exposures exploits an approximated analytical technique which applies to one-factor Merton type models. This method, originally introduced by Vasicek [26], consists in replacing the original portfolio loss distribution with an asymptotic one, whose VaR can be computed analytically. This approach is also known as the Asymptotic Single-Risk Factor (ASRF) paradigm which underlies the Internal Ratings-Based (IRB) framework of the Pillar I of Basel II [20] regulation, and it is an example of a very simplified model for credit VaR. Basic hypothesis of this model include the homogeneity of the underlying portfolio and a common factor driving systematic risk. As such, the model does not consider concentration, sector and contagion effects, but it allows for an easy and fast computation of the credit portfolio VaR.

The regulator is aware of the severe flaws of the model and considers its results a (regulatory) minimal, albeit (economically) insufficient capital at risk, and it forces financial institutions to equip themselves with more sophisticated tools to account for the aforementioned neglected risks. Ideally, these sophisticated tools should be an extension to the simplified model, so as to make the comparison with the latter straightforward, and they should retain the analytical feature so as to lighten as much as possible the computation burden.

According to this criterion, the difference between true and asymptotic VaR has been computed analytically through a second order approximation [14]. Many steps have been taken in this direction, extending the original Vasicek result for homogeneous portfolios to include granularity risk [28], [18], [13], [8] and sectoral concentration risk (see Pykhtin [21]). In these works, concentration risk represents a violation of the ASFR model and can be decomposed into two contributions: an idiosyncratic part, *single name* or *imperfect granularity risk*, due to the small size of the portfolio or to the presence of large exposures associated to single obligors and a systematic term, *sectoral concentration*, due to imperfect diversification across sectorial factors.

The third source of risk, *credit contagion*, lies somewhat in-between the previous two [20]. This risk takes into account the occurrence of default events triggered by inter-dependencies (legal, financial, business-oriented) among obligors. Very diverse approaches have been proposed to tackle this problem. Davis and Lo [5] have built a first model where the default of any company in the portfolio can infect all the others. Egloff *et al* [7] have developed a neural-network inspired model to mimic the structure of links among obligors in a portfolio. Recently, Rösch *et al* [22] have proposed an extension of [5] in a default mode scenario where obligors are divided into two categories: those who can be considered immune from contagion "I-firms" (infecting) and those who can be contaminated "C-firms" .

In this paper, we study in a unified framework the effects of concentration and contagion risk. A first attempt to generalize the work by Pykhtin in this direction has been pursued by Yun [30]. However, the resulting model specification appears incomplete to some extent and hardly applicable to concrete problems. A more complete and detailed setting has been presented in Bonollo *et al* [3], who combine Pykhtin's idea with the contagion specification proposed by Rösch [22], obtaining a model that is general enough and yet preserves analytical tractability. Here, we push forward the analysis and try and overcome one of the flaws in this model, namely the static probabilities of default (PD) and recovery rates independent of the PDs. Actually, these deficiencies are common also to other analytical models, such as CreditRisk+ [27] developed by Credit Suisse, and only more computationally intensive frameworks, such as CreditMetrics [16], enjoy richer specifications as for defaults and recovery rates, typically via a rating migration's modeling. Recently some works have introduced a dependence between the PDs and the recovery rates within the ASRF model (see for example Sanchez *et al* [23] and Sen [24]), although they do not consider a multi-factor economy and contagion amongst obligors. The model we present in what follows allows for stochastic probabilities of default and recovery rates, while retaining the benefits of the analytical computation, thus combining the desirable features of several renowned models. Besides, we show that it can be considered under different perspectives, even a rating migration one, so that it allows for a flexible specification to best suit the financial institution's needs.

The paper is organized as follows. In Section 2 we present and motivate the main idea, based on the introduction of scenarios. Section 3 is devoted to the description of the model, including a thorough specification of the contagion part. The main theoretical results, expressing the value at risk of a credit portfolio in analytical terms, are exposed in Section 4. Section 5 concludes with a detailed numerical analysis. Section 6 summarizes and collects our final remarks. Technical details and issues can be found in the Appendix.

2 General framework: scenarios

Before giving a detailed description of the model we highlight some of its underlying assumptions. The starting point is a multi-factor default mode Merton model. Default happens when a continuous variable X_i describing the financial well-being of borrower i at the horizon falls below a certain threshold. As mentioned in the Introduction, an extension of the multi-factor model including the effects of contagion risk has already been proposed in [3]. Here we focus on the generalization which allows the rating features of each obligor to loose their static character. More precisely, we relax the assumption of constant probability of default and loss given default, for each borrower during the chosen time horizon, and allow them to assume values, randomly drawn from a finite distribution. This is achieved through the introduction of different possible scenarios.

In the model's specification that we present in Section 3, each obligor can in theory be assigned different probabilities of default and losses given default, thus identifying a corresponding number of scenarios; the total number of scenarios will be the sum of all the obligors' specific scenarios, which could result to be rather large. In practice, though, obligors are gathered, on the basis of their creditworthiness, in a certain number of rating classes each featured by its own PD and LGD: in this case it is possible to dramatically reduce the number of possible scenarios under some assumptions. Actually, in the numerical analysis we show in Section 5 we use this re-casting of the obligors into a predefined set of rating classes, which is a standard approach in the banking industry. This, besides making the number of scenarios practically manageable, allows also to interpret a scenario under two economically and financially meaningful perspectives:

1. As a given state of the economy, with its specific probabilities of default and losses given default (or, equivalently, recovery rates) associated to each rating class. In each scenario, the obligors always belong to the same rating class, but rating classes' PDs and LGDs change with respect to other scenarios, due to cyclical conditions of the economy. As an example, a recession can give rise to a scenario with generally higher PDs and LGDs than a growth period.
2. As a given state of economy, with its rating classes containing a given set of obligors. In this case, for each rating class, PDs and LGDs are constant through all the possible scenarios, but in each scenario the composition (in terms of obligors) of each single rating class is different. In practice, it is as if we were modeling the rating migration of the obligors.

Either perspectives above can be expressed by means of the following general setting of the loan's portfolio:

- The loans are associated to M distinct borrowers. Each borrower has exactly one loan characterized by exposure EAD_i. We define the weight of a loan in the portfolio as $w_i = EAD_i / \sum_{i=1}^{M} EAD_i$
- The uncertainty on the rating features of the ith borrower is modeled through the introduction of S scenarios, each characterized by possible values which can be assumed by the default probability PD_i and the loss-given default $LGD_i = Q_i$ (where Q stands for a stochastic variable, with mean μ and standard deviation σ):

$$(PD, Q)_i = \begin{cases} (p_1, Q_1)_i & \text{with probability } \lambda_{i1} \\ (p_2, Q_2)_i & \text{with probability } \lambda_{i2} \\ \vdots & \vdots \\ (p_S, Q_S)_i & \text{with probability } \lambda_{iS} \end{cases} \quad (1)$$

  where $\sum_{\varphi=1}^{S} \lambda_{i\varphi} = 1$ for each i. Scenarios are independent of each others and each Q is assumed to be independent of the the other Qs and the remaining stochastic variables of the model.

This setting includes the possibility to introduce rating classes, so that all the obligors belonging to one of those, have the same PD and LGD. If we build scenarios according to the first perspective, in each of them the number of PDs and LGDs is reduced from M to the number of rating classes (usually below 20). When we build scenarios according to the second perspective, we cannot really abate substantially the number of scenarios unless we make strong assumptions on the possible migrations of the single obligors. An example of such assumptions is shown in Section 5.

Finally, it is worth mentioning the fact that the framework allows for an (implicit) correlation between the level of the default probabilities and the losses given default, simply by devising scenarios where higher LGDs are associated to higher PDs. This is an interesting feature that takes into account an effect widely observed in practice and extensively documented in the literature (see Sironi *et al* [25]). To our knowledge, such effect is usually neglected in the most popular models, although some models presented in theoretical works account for that (an incomplete list includes Merton [19], Black and Cox [1] and, more recently, Frye [9] and [10], Jarrow [17], Carey and Gordy [4]).

3 Model specification

We now turn to the description of the theoretical model. Since we start to develop it from a multi-factor environment, we choose to follow the notation already adopted in [3] and originally introduced by Pykhtin [21]. First we present the multi-factor specification, secondly we add the contagion part.

3.1 Multi-factor setup

Asset returns $\{X_i\}_{i=1,\ldots,M}$ are the key variables to be modeled: default occurs for borrower i, in a given scenario φ, when the corresponding $X_{i\varphi}$ falls below the threshold $N^{-1}(p_{i\varphi})$. Asset returns are assumed to be distributed according to a standard normal distribution:

$$X_{i\varphi} = r_{i\varphi} Y_i + \sqrt{1 - r_{i\varphi}^2}\, \xi_i \,, \tag{2}$$

where the systematic contribution is expressed in terms of a composite variable $\{Y_i\}_{i=1\ldots M}$, encoding the effects of multiple sectors (see Appendix A1) and the idiosyncratic component of risk, which can be diversified away in the case of an infinitely granular portfolio, is given by $\xi_i \sim \mathcal{N}(0,1)$ independent of Y_i. In the most general case, we let the sensitivity of borrower i to systematic risk, namely $r_{i\varphi} \geq 0$, depend on the given scenario.

The composite factor can be expressed as a linear combination of N independent systematic factors $Z_k \sim \mathcal{N}(0,1)$, $k = 1, \ldots, N$,

$$Y_i = \sum_{k=1}^{N} \alpha_{ik}\, Z_k \,, \tag{3}$$

the assumption of unit variance yielding $\sum_{k=1}^{N} \alpha_{ik}^2 = 1$. In turn, the quantity $r_{i\varphi} Y_i$ can be rewritten in terms of

- a *unique* systematic risk factor $\overline{Y} = \sum_{k=1}^{N} b_k Z_k$, with $b_k \geq 0$,
- a residual contribution $\sum_{k=1}^{N} (r_{i\varphi}\alpha_{ik} - a_{i\varphi} b_k) Z_k$, independent of $\overline{Y}$, which encodes the conditional asset correlation[1]

$$\rho^{Y}_{i\varphi,j\psi} = \frac{r_{i\varphi} r_{j\psi} \sum_{k=1}^{N} \alpha_{ik}\alpha_{jk} - a_{i\varphi} a_{j\psi}}{\sqrt{(1 - a_{i\varphi}^2)(1 - a_{j\psi}^2)}} \,. \tag{4}$$

[1] The unconditional correlation between borrowers i and j, respectively in scenarios φ and ψ, is given by $\mathrm{corr}(X_{i\varphi}, X_{j\psi}) \equiv \rho_{i\varphi,j\psi} = r_{i\varphi} r_{j\psi} \sum_{k=1}^{N} \alpha_{ik}\, \alpha_{jk}$.

The non-negative coefficients $a_{i\varphi} \equiv r_{i\varphi} \sum_{k=1}^{N} \alpha_{ik} b_k$, are effective factor loadings, obtainable through an optimization procedure, sketched in Appendix B. The unit variance constraint enforces $\sum_{k=1}^{N} b_k^2 = 1$.

Therefore, asset returns for obligor i, in a given scenario φ, can be cast into the general form

$$X_{i\varphi} = a_{i\varphi}\overline{Y} + \sum_{k=1}^{N} (r_{i\varphi}\alpha_{ik} - a_{i\varphi}b_k)\, Z_k + \sqrt{1 - r_{i\varphi}^2}\, \xi_i \,. \tag{5}$$

3.2 Credit contagion

Contagion risk can be ascribed to inter-company ties, such as legal (parent-subsidiary) relationships, financial and business oriented relations (supplier-purchaser interactions) and so on. This entails a complex network of links among obligors, which makes the credit contagion problem very hard to solve.

Here we adopt a simplified perspective, already exposed in Bonollo *et al* [3]. We assume that obligors are broadly divided into two categories: those firms which are immune from contagion (referred to as "I-firms", i.e. infecting) and those companies which can be contaminated by the first group through credit contagion ("C-firms"). Asset returns associated to group "I" follow the multi-factor specification given by eq. (2) while "C-firms"' asset returns (for obligor i in a scenario φ) are assumed to satisfy

$$X_{i\varphi} = r_{i\varphi} Y_i + \sqrt{1 - r_{i\varphi}^2}\, \xi(\Gamma_i, \epsilon_i) \,. \tag{6}$$

The firm-specific factor $\xi(\Gamma_i, \epsilon_i)$, which is assumed to be scenario-independent, is defined by

$$\xi(\Gamma_i, \epsilon_i) = g_i \Gamma_i + \sqrt{1 - g_i^2}\, \epsilon_i \tag{7}$$

where

- ϵ_i is the usual idiosyncratic contribution,
- the term $g_i \Gamma_i$ encodes the effects of contagion risk. The composite contagion factor Γ_i can be written as a sum over latent contagion variables C_k (assumed to be independent and distributed as $\mathcal{N}(0,1)$)

$$\Gamma_i = \sum_{k=1}^{N} \gamma_{ik}\, C_k \,.$$

The unit variance property of $X_{i\varphi}$ is preserved if $\sum_{k=1}^{N} \gamma_{ik}^2 = 1$. We decompose each sector into a "I" segment and a "C" one. Therefore, the contagion effect experienced by an arbitrary "C-firm" can be thought of as the weighted sum of contributions coming from the infecting segments of different sectors. Under this specification, the number of latent contagion factors equals the number of industry-geographic factors, N. The coefficient g_i plays the role of a contagion factor loading and represents a measure of how much obligor i is overall affected by contagion. It is worth noticing that eq.s (6-7) express in compact form also the behavior of "I-firms", with the understanding that $g_i = 0$ in that case. We will come back to the estimation of the contagion parameters in the Appendix A2.

3.3 Summary

Having specified the model in this way, by making the effects due to multi-factors and contagion explicit, asset returns, for obligor i in a given scenario φ, then follow

$$\begin{aligned} X_{i\varphi} &= a_{i\varphi}\overline{Y} + \sum_{k=1}^{N}(r_{i\varphi}\alpha_{ik} - a_{i\varphi}b_k)Z_k + \\ &+ \sqrt{1-r_{i\varphi}^2}\, g_i \sum_{k=1}^{N} \gamma_{ik}C_k + \\ &+ \sqrt{1-r_{i\varphi}^2}\sqrt{1-g_i^2}\,\epsilon_i\,. \end{aligned} \tag{8}$$

The conditional correlation between distinct obligors i and j, respectively in scenarios φ and ψ, assumes the form

$$\rho^{YC}_{i\varphi,j\psi} = \frac{r_{i\varphi}r_{j\psi}\sum_{k=1}^{N}\alpha_{ik}\alpha_{jk} + \sqrt{1-r_{i\varphi}^2}\sqrt{1-r_{j\psi}^2}\, g_i g_j \sum_{k=1}^{N}\gamma_{ik}\gamma_{jk} - a_{i\varphi}a_{j\psi}}{\sqrt{(1-a_{i\varphi}^2)(1-a_{j\psi}^2)}}\,. \tag{9}$$

4 VaR decomposition and results

Given this setup, the portfolio loss rate L can be written as the weighted sum over individual loss rates

$$L = \sum_{i=1}^{M} w_i L_i\,.$$

Each L_i is a stochastic variable whose value is allowed to vary across different scenarios

$$L_i = \begin{cases} Q_{i1}\,\mathbf{1}_{\{X_{i1}\le N^{-1}(p_{i1})\}} & \text{with probability } \lambda_{i1} \\ Q_{i2}\,\mathbf{1}_{\{X_{i2}\le N^{-1}(p_{i2})\}} & \text{with probability } \lambda_{i2} \\ \vdots & \vdots \\ Q_{iS}\,\mathbf{1}_{\{X_{iS}\le N^{-1}(p_{iS})\}} & \text{with probability } \lambda_{iS} \end{cases} \tag{10}$$

$Q_{i\varphi}$ and the indicator function $\mathbf{1}_{\{.\}}$ represent respectively the stochastic LGD and the event of default, associated to obligor i, in a given scenario. Our goal is to calculate the quantile at confidence level q of this quantity, namely $t_q(L)$. Here, we sketch briefly the main steps of the calculation, following the notation adopted by Pykhtin [21], thanks to the original contributions developed by [18] and [14].

The main idea consists in calculating $t_q(L)$ analytically, through a Taylor expansion around the quantile of another variable $\overline{L}$, such that $t_q(\overline{L})$ is analytically tractable and sufficiently close to $t_q(L)$. Therefore, the portfolio loss L can be expressed in terms of a new variable $\overline{L}$

$$L \equiv \overline{L} + U \,,$$

where $U = L - \overline{L}$ plays the role of a perturbation. Rendering explicit the dependence of L on the scale of the perturbation, we can write

$$L_\varepsilon \equiv \overline{L} + \varepsilon U$$

with the understanding that the original definition of L is recovered for $\varepsilon = 1$. The key result obtained in [18] allows to compute, for high enough confidence level q, the quantile $t_q(L_\varepsilon)$ as a series expansion in powers of ε around $t_q(\overline{L})$. Up to the second order, $t_q(L) \equiv t_q(L_{\varepsilon=1})$ reads

$$t_q(L) \approx t_q(\overline{L}) + \left.\frac{dt_q(L_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} + \frac{1}{2}\left.\frac{d^2 t_q(L_\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} . \tag{11}$$

Each term of this expansion will be given a separate analysis. While we leave the thorough discussion of the zeroth order term $t_q(\overline{L})$ to the next section, here we will spend a few words about higher order contributions.

The first and second derivatives of VaR have been originally calculated by Gourieroux *et al.* [14]. Their expressions are given by

$$\left.\frac{dt_q(L_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = E[U|\overline{L} = t_q(\overline{L})] \,, \tag{12}$$

$$\left.\frac{d^2 t_q(L_\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} = -\frac{1}{f_{\overline{L}}(l)} \frac{d}{dl}\left.\left(f_{\overline{L}}(l)\,\mathrm{var}[U|\overline{L} = l]\right)\right|_{l=t_q(\overline{L})} , \tag{13}$$

where $f_{\overline{L}}(.)$ is the probability density function of $\overline{L}$ and $\mathrm{var}[U|\overline{L} = l]$ is the variance of U conditional on $\overline{L} = l$.

The key point now consists in choosing the appropriate $\overline{L}$. We follow the path traced by Pykhtin [21] in the multi-sector case, extending it in order to include different scenarios, besides contagion already introduced in [3]. We anticipate that, with a proper choice of $\overline{L}$ (also the most intuitively appealing), the first order correction can be caused to vanish, for any confidence level q.

4.1 Zeroth order term

We define the variable $\overline{L}$ as the limiting loss distribution in the one-factor Merton framework [19] i.e.

$$\overline{L} \equiv l(\overline{Y}) = E[L|\overline{Y}] \,. \tag{14}$$

Performing the calculations explicitly, we get

$$
\begin{aligned}
\overline{L} &\equiv E\left[\sum_{i=1}^{M} w_i\, Q_i\, \mathbf{1}_{\{X_i \leq N^{-1}(p_i)\}} \middle| \overline{Y}\right] = \\
&= \sum_{i=1}^{M} w_i \sum_{\varphi=1}^{S} \lambda_{i\varphi}\mu_{i\varphi}\, \mathbb{P}(X_{i\varphi} \leq N^{-1}(p_{i\varphi})|\overline{Y}) = \\
&= \sum_{i=1}^{M} w_i \sum_{\varphi=1}^{S} \lambda_{i\varphi}\mu_{i\varphi}\, N\left[\frac{N^{-1}(p_{i\varphi}) - a_{i\varphi}\overline{Y}}{\sqrt{1-a_{i\varphi}^2}}\right] = \\
&= \sum_{i=1}^{M} w_i \sum_{\varphi=1}^{S} \lambda_{i\varphi}\mu_{i\varphi}\, \hat{p}_{i\varphi}(\overline{Y})\,, \qquad (15)
\end{aligned}
$$

where $\hat{p}_{i\varphi}(y)$ is the probability of default of borrower i, given scenario φ, conditional on $\overline{Y} = y$:

$$
\hat{p}_{i\varphi}(y) = N\left[\frac{N^{-1}(p_{i\varphi}) - a_{i\varphi}y}{\sqrt{1-a_{i\varphi}^2}}\right].
$$

(N indicates the cumulative normal distribution). The quantile of $\overline{L}$ at level q can be calculated analytically as

$$
t_q(\overline{L}) = l(N^{-1}(1-q))\,. \qquad (16)
$$

The complete proof is presented in Appendix C1.

4.2 First order term

The first order derivative (12) of VaR is expressed as the expectation of $U = L - \overline{L}$, conditional on $t_q(\overline{L}) = l$. Given the discussion in Appendix C1 about the monotonicity and inversion properties of $\overline{L} = l(\overline{Y})$, such a conditioning is equivalent to $\overline{Y} = y = N^{-1}(1-q)$. Therefore, given the definition of $\overline{L}$ in (14), the first derivative becomes

$$
\left.\frac{dt_q(L_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = E[U|\overline{Y} = N^{-1}(1-q)]
$$

and vanishes automatically.

4.3 Second order term

We are finally left with the second order contribution, containing the term (13). Recalling the observation about conditioning developed in the previous paragraph, the second order derivative yields

$$
\left.\frac{d^2 t_q(L_\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} = -\frac{1}{n(y)}\frac{d}{dy}\left(n(y)\frac{\nu(y)}{l'(y)}\right)\Bigg|_{y=N^{-1}(1-q)},
$$

where $\nu(y) \equiv \mathrm{var}(U|\overline{Y} = y)$ is the conditional variance of U, $l'(.)$ is the first derivative of $l(.)$ and $n(.)$ is the standard normal density. By carrying out the derivative with respect to y explicitly and using the fact that $n'(y) = -y\,n(y)$, it turns out [21]

$$\left.\frac{d^2 t_q(L_\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} = -\frac{1}{l'(y)}\left[\nu'(y) - \nu(y)\left(\frac{l''(y)}{l'(y)} + y\right)\right]\Bigg|_{y=N^{-1}(1-q)}$$

4.4 The complete formula

Pulling all the pieces together and considering that first-order contributions cancel out, the total approximated VaR up to second order is given by

$$t_q(L) \approx t_q(\overline{L}) + \Delta t_q \tag{17}$$

where

$$\Delta t_q = -\frac{1}{2\,l'(y)}\left[\nu'(y) - \nu(y)\left(\frac{l''(y)}{l'(y)} + y\right)\right]\Bigg|_{y=N^{-1}(1-q)} . \tag{18}$$

The function $l(y)$ is defined as in (14), (15), and $\nu(y) = \mathrm{var}[L|\overline{Y} = y]$ is the conditional variance of L on $\overline{Y} = y$.

$\nu(y)$ (and hence the total correction) can be further decomposed in terms of its systematic and idiosyncratic components

$$\nu(y) = \nu_\infty(y) + \nu_{GA}(y)\,, \tag{19}$$

where

$$\begin{aligned} \nu_\infty(y) &= \mathrm{var}[E(L|\{Z_k\})|\overline{Y} = y]\,, \\ \nu_{GA}(y) &= E[\mathrm{var}(L|\{Z_k\})|\overline{Y} = y]\,. \end{aligned} \tag{20}$$

Formulae (17) through (20) encodes the effects of concentration risk[2]:

- sector concentration affects both the zeroth order term $t_q(\overline{L})$, in an implicit way and by construction, and the second order correction depending on $\nu_\infty(y)$. The latter, obtained in the limit of an infinitely fine-grained portfolio, represents the systematic component of risk which cannot be diversified away;
- single name concentration is described by the granularity adjustment term which comes from eq. (18), whenever $\nu(y) = \nu_{GA}(y)$. For a large enough number of obligors M (ideally, in the limit $M \to \infty$) and under the condition of a sufficiently homogeneous distribution of loans' exposures (in mathematical terms $\sum_{i=1}^M w_i^2 \to 0$, while $\sum_{i=1}^M w_i = 1$) the granularity contribution vanishes;
- contagion risk shows its effects on the second order correction only. However, unlike granularity and sector concentration, it is not an additive-type contribution but affects implicitly the conditional variance $\nu(y)$ as a whole, through the conditional correlation matrix given by formula (9), which encodes synthetically all the information about contagion (see final formulas).

[2]Explicit expressions for the derivatives of $l(y)$ and $\nu(y)$ can be found in Appendix C2.

Since we have already provided an analytical formula for the zeroth order term, eq. (16), the final step consists in giving explicit expressions for the corrections (20). It turns out they are more easily calculated starting from the very definition of the conditional variance:

$$\begin{aligned} \nu(y) &\equiv \operatorname{var}[L|\overline{Y}=y] = \operatorname{var}\left[\sum_{i=1}^{M} w_i\, L_i \middle| \overline{Y}=y\right] = \\ &= \sum_{i=1}^{M} w_i^2 \operatorname{var}[L_i|\overline{Y}=y] + \sum_{i\neq j=1}^{M} w_i w_j \operatorname{cov}[L_i, L_j|\overline{Y}=y]\,. \end{aligned} \tag{21}$$

We proceed to compute the two contributions on the right hand side separately.

- The variance term can be expressed as

$$\operatorname{var}[L_i|\overline{Y}] = E[L_i^2|\overline{Y}] - E[L_i|\overline{Y}]^2\,.$$

  Given the previous definition of L_i, eq. (10), and

$$L_i^2 = \begin{cases} Q_{i1}^2\, \mathbf{1}_{\{X_{i1}\leq N^{-1}(p_{i1})\}} & \text{with probability } \lambda_{i1} \\ Q_{i2}^2\, \mathbf{1}_{\{X_{i2}\leq N^{-1}(p_{i2})\}} & \text{with probability } \lambda_{i2} \\ \vdots & \vdots \\ Q_{iS}^2\, \mathbf{1}_{\{X_{iS}\leq N^{-1}(p_{iS})\}} & \text{with probability } \lambda_{iS} \end{cases} \tag{22}$$

  where we have used the property $\mathbf{1}_{\{.\}}^2 = \mathbf{1}_{\{.\}}$ of the indicator function, we get

$$\begin{aligned} E[L_i^2|\overline{Y}=y] &= \sum_{\varphi=1}^{S} \lambda_{i\varphi}(\mu_{i\varphi}^2 + \sigma_{i\varphi}^2)\, \hat{p}_{i\varphi}(y)\,, \\ E[L_i|\overline{Y}=y]^2 &= \sum_{\varphi=1}^{S}\sum_{\psi=1}^{S} \lambda_{i\varphi}\lambda_{i\psi}\mu_{i\varphi}\mu_{i\psi}\, \hat{p}_{i\varphi}(y)\hat{p}_{i\psi}(y)\,, \end{aligned}$$

  which lead to

$$\operatorname{var}[L_i|\overline{Y}=y] = \sum_{\varphi=1}^{S} \lambda_{i\varphi}\, \hat{p}_{i\varphi}(y) \left[(\mu_{i\varphi}^2 + \sigma_{i\varphi}^2) - \mu_{i\varphi} \sum_{\psi=1}^{S} \lambda_{i\psi}\mu_{i\psi}\, \hat{p}_{i\psi}(y)\right]\,. \tag{23}$$

- Similarly, the covariance term can be decomposed into

$$\operatorname{cov}[L_i, L_j|\overline{Y}] = E[L_i L_j|\overline{Y}] - E[L_i|\overline{Y}]E[L_j|\overline{Y}]\,.$$

  The first component yields

$$E[L_i L_j|\overline{Y}=y] = \sum_{\varphi=1}^{S} \mu_{i\varphi} \sum_{\psi=1}^{S} \mu_{j\psi}\, \lambda_{i\varphi,j\psi}\, N_2(N^{-1}(\hat{p}_{i\varphi}(y)), N^{-1}(\hat{p}_{j\psi}(y)), \rho_{i\varphi,j\psi}^{YC})\,,$$

  where $N_2(.,.,.)$ is the bivariate normal cumulative distribution function and $\lambda_{i\varphi,j\psi}$ represents the joint probability that obligor i assumes values in scenario

φ and obligor j in scenario ψ. The other component of the covariance reads explicitly as

$$E[L_i|\overline{Y}=y]E[L_j|\overline{Y}=y] = \sum_{\varphi=1}^{S} \lambda_{i\varphi}\mu_{i\varphi}\,\hat{p}_{i\varphi}(y) \sum_{\psi=1}^{S} \lambda_{j\psi}\mu_{j\psi}\,\hat{p}_{j\psi}(y)\,.$$

Therefore, the final result is

$$\begin{aligned}\operatorname{cov}[L_i, L_j|\overline{Y}=y] = &\sum_{\varphi=1}^{S}\sum_{\psi=1}^{S} \mu_{i\varphi}\mu_{j\psi}\,\lambda_{i\varphi,j\psi}\,N_2(N^{-1}(\hat{p}_{i\varphi}(y)), N^{-1}(\hat{p}_{j\psi}(y)), \rho^{YC}_{i\varphi,j\psi}) \\ &-\sum_{\varphi=1}^{S}\sum_{\psi=1}^{S} \mu_{i\varphi}\mu_{j\psi}\,\lambda_{i\varphi}\lambda_{i\psi}\,\hat{p}_{i\varphi}(y)\,\hat{p}_{j\psi}(y)\,.\end{aligned} \tag{24}$$

Coming back to the original problem of calculating the conditional variance of the total loss distribution L, we plug eq.s (23) and (24) into eq. (21). Adding and subtracting the contribution corresponding to $i=j$ in the covariance sum, eq. (21) can be eventually decomposed in terms of its systematic and idiosyncratic components, namely:

$$\begin{aligned}\nu_\infty(y) \;=\; &\sum_{i,j=1}^{M} w_i w_j \sum_{\varphi=1}^{S}\sum_{\psi=1}^{S} \mu_{i\varphi}\mu_{j\psi}\,\lambda_{i\varphi,j\psi}\,N_2(N^{-1}(\hat{p}_{i\varphi}(y)), N^{-1}(\hat{p}_{j\psi}(y)), \rho^{YC}_{i\varphi,j\psi}) \\ &-\sum_{i,j=1}^{M} w_i w_j \sum_{\varphi=1}^{S}\sum_{\psi=1}^{S} \mu_{i\varphi}\mu_{j\psi}\,\lambda_{i\varphi}\lambda_{i\psi}\,\hat{p}_{i\varphi}(y)\,\hat{p}_{j\psi}(y)\,,\end{aligned} \tag{25}$$

and

$$\begin{aligned}\nu_{GA}(y) \;=\; &\sum_{i=1}^{M} w_i^2 \sum_{\varphi=1}^{S} \lambda_{i\varphi}(\mu_{i\varphi}^2+\sigma_{i\varphi}^2)\,\hat{p}_{i\varphi}(y) \qquad (26) \\ &-\sum_{i=1}^{M} w_i^2 \sum_{\varphi,\psi=1}^{S} \lambda_{i\varphi,i\psi}\,\mu_{i\varphi}\mu_{i\psi}\,N_2(N^{-1}(\hat{p}_{i\varphi}(y)), N^{-1}(\hat{p}_{i\psi}(y)), \rho^{YC}_{i\varphi,i\psi})\,.\end{aligned}$$

The conditional correlation appearing in eq.s (25), (26) is given by formula (9).

4.5 Summary of the results

The final result can be stated as follows: the quantile at level q of the loss distribution L, $t_q(L)$, is given by the approximated formula (17), where

- the asymptotic zeroth order term $t_q(\overline{L})$ is expressed by eq.s (16), (15)
- the total correction Δt_q is encoded into eq.s (18), (19), (25) and (26).

5 Numerical analysis

This section is devoted to the numerical implementation of the theoretical model. An in-depth analysis of the behavior of the approximated value at risk as a function of the numbers of obligors M and the rating quality of the portfolio has already been presented in [3]. Those results, though referring to the single scenario case, continue to hold in the presence of multiple scenarios. Here, we put the focus on the effects of contagion and different scenarios.

5.1 Portfolio data and parameters of the model

Before entering the details, some general information about the characterization of the portfolio and the model itself is given, while a detailed description of the parameters entering the model as inputs and their relationship to observable data are thoroughly exposed in Appendix A, to which we will refer in the following.

- Loan exposures are assigned following the empirical rule, described for example in [13], namely $EAD_i = (i^3)$. Such a power law yields a sufficiently granular portfolio, though ensuring that for $M \sim 100$ and above, the loan to one borrower limit of 4% of the total portfolio size is not exceeded.
- We sort obligors in ascending order with respect to their exposure. We further assume that the last 20% of them belongs to the group of infecting "I-firms".
- We consider $N = 11$ industry-geographic sectors based on the GICS classification scheme for sector activities. We assume them to be standardized but dependent on each others through an appropriate correlation matrix derived from MSCI EMU industry indices [3].

  To make things simpler, we assume that each obligor is associated only with one sector. The criterion with which this correspondence is established leads to different distributions of the portfolio loans onto sectors. In the following, we generally assume a homogeneous distribution, devoting a separate paragraph to the analysis of sector concentration.

  Given this setup, it is straightforward to express each obligor's asset return in terms of the independent standardized normal risk factors $\{Z_k\}$. The dependence of each obligor on these variables is expressed by two parameters:

  - the α_{ik} coefficients,
  - the factor loading $r_{i\varphi}$,

  whose description can be found in Appendix A1.
- Similarly, for the contagion specification we use the same structure of sectors. What changes with respect to the plain multi-factor setup is:
  - The definition and the values of the "participation" coefficients δ_{ik}^{C}, measuring how much a company is affected by the infecting segment of a given sector (see Appendix A2).

[3] For details, see [6] and [15].

– The contagion factor loading g_i, which is a discretionary parameter.
– We generally assume that only two contagion sectors affect each obligor. This simplification will be removed in the last paragraph where we explicitly study the effects of contagion (a sort of concentration) on the value at risk.

In order to lighten the computational burden, we classify both δ^C_{ik} and g_i into buckets, e.g. three for the former and one for the latter. Finally, following Appendix A2, we can express the contagion part of asset returns in terms of the independent latent variables $C_k \sim \mathcal{N}(0,1)$, through the coefficients γ_{ik}.

- The rating quality of the portfolio is directly related to the presence of different scenarios. To simplify the subsequent numerical analysis, obligors are grouped into rating classes.

  Adopting the specification already employed in [3], based on the subdivision into 7 rating classes that was originally proposed by Gordy [12], we start from a particular realization of obligors aggregation (denoted by the superscript "0"), which will also be used in the next sections to build new scenarios. Its properties are listed in Table 1.

	AAA	AA	A	BBB	BB	B	CCC
$PD \equiv p^0$	0.01%	0.02%	0.06%	0.18%	1.06%	4.94%	19.14%
$\mu \equiv \mu^0$	10%	20%	25%	30%	35%	40%	50%

Table 1: Rating classes according to Gordy [12]

  We have chosen the LGD mean values so as to have on average $\mu^0 = 30\%$. The corresponding standard deviations are set to $\sigma^0 = 1/2\sqrt{\mu^0(1-\mu^0)}$.

  For notational purposes we group the PD and LGD values into vectors of 7 elements:

$$\begin{aligned} p^0 &= (p^0_{AAA}, p^0_{AA}, \ldots, p^0_{CCC}) \\ \mu^0 &= (\mu^0_{AAA}, \mu^0_{AA}, \ldots, \mu^0_{CCC}) \\ \sigma^0 &= (\sigma^0_{AAA}, \sigma^0_{AA}, \ldots, \sigma^0_{CCC}) \end{aligned}$$

We now address the core discussion of the numerical analysis. We will be interested in studying the behavior of the following quantities:

- $t_q(L)$ = second order approximated VaR;
- $t_q(\bar{L})$ = zeroth order, asymptotic, VaR;
- Δ_{YC} = total correction due to multi-factoriality and contagion;
- Δ_C = total correction due to contagion;
- Δ_{GA} = granularity adjustment;
- Δ_∞ = correction due to multi-sectoriality and contagion, for a portfolio homogeneous in the exposures;

5.2 Scenario specification

As we mentioned above, the scenario framework is rather flexible and naturally leads to different interpretations. Two of them have been sketched in Section 2 and here we push the analysis in more depth, by dealing with scenario specifications that are suitable for the numerical analysis, object of the next paragraphs.

Scenarios and numerical implementations

In the following analysis, we assume that obligors are *perfectly* correlated with each others. Therefore, the behavior of a single obligor i in terms of his evolution towards a scenario φ describes as well the behavior of all the other obligors. Expressing this concept in terms of the joint probability $\lambda_{i\varphi,j\psi}$ we obtain the relation

$$\lambda_{i\varphi,j\psi} = \lambda_\varphi \delta_{\varphi,\psi}\,, \tag{27}$$

where λ_φ is the probability weight assigned to scenario φ (independent of the obligors) and $\delta_{\varphi,\psi}$ stands for the Kronecker's delta. Specifying eq.s (25) and (26) in terms of the new joint probability (27), double sums reduce to single ones. We obtain:

$$\begin{aligned} \nu_\infty(y) \;=\; & \sum_{i,j=1}^{M} w_i w_j \sum_{\varphi=1}^{S} \mu_{i\varphi}\mu_{j\varphi}\, \lambda_\varphi\, N_2(N^{-1}(\hat{p}_{i\varphi}(y)), N^{-1}(\hat{p}_{j\varphi}(y)), \rho^{YC}_{i\varphi,j\varphi}) \\ & - \sum_{i,j=1}^{M} w_i w_j \sum_{\varphi=1}^{S}\sum_{\psi=1}^{S} \mu_{i\varphi}\mu_{j\psi}\, \lambda_{i\varphi}\lambda_{i\psi}\, \hat{p}_{i\varphi}(y)\, \hat{p}_{j\psi}(y)\,, \end{aligned} \tag{28}$$

and

$$\begin{aligned} \nu_{GA}(y) \;=\; & \sum_{i=1}^{M} w_i^2 \sum_{\varphi=1}^{S} \lambda_{i\varphi}(\mu_{i\varphi}^2 + \sigma_{i\varphi}^2)\, \hat{p}_{i\varphi}(y) \\ & - \sum_{i=1}^{M} w_i^2 \sum_{\varphi=1}^{S} \lambda_\varphi\, \mu_{i\varphi}^2\, N_2(N^{-1}(\hat{p}_{i\varphi}(y)), N^{-1}(\hat{p}_{i\varphi}(y)), \rho^{YC}_{i\varphi,i\varphi})\,. \end{aligned} \tag{29}$$

Analogously, formulae in Appendix C2 drop the double sums on scenarios and assume a simpler form.

As already stated in the general theoretical framework, the model deals with scenarios defined for each single obligor. However, following the common practice, we simplify the problem by aggregating obligors into rating classes, thus effectively analyzing *joint* scenarios, referring to whole classes of rating: this significantly lightens the computational burden, without being an unrealistic choice. In addition, in the numerical implementation we limit ourselves to a relatively small number of scenarios, namely up to three.

We propose two different implementations and besides, for comparison's purposes, we introduce also the special case when only one scenario is present, characterized by the vectors (p^0, μ^0, σ^0) and probability weight $\lambda_1 = 1$. The distribution of obligors onto the different rating classes (i.e. the elements of (p^0, μ^0, σ^0)) is assumed to be that of an *average quality* portfolio, such that speculative grade loans account for 50% of the total exposure. We label this single scenario case as "Sgl".

As for the other two implementations, we adhere to the two ways of looking at the scenarios' building already introduced in Section 2. In more details:

1. We choose different scenarios where, in each of them, we increment/decrement the features of all ratings (PDs and LGDs) of some percentage amount, reflecting a worsening/improvement of the general economic situation.

   In this picture, obligors do not move from their originally assigned rating class;

2. We interpret scenarios in terms of a particularly simple kind of migration. As initial state, at time $t = 0$, we choose the single scenario setup. Changes occur at time $t = 0^+$ and can be seen as joint migrations of obligors towards other rating classes. This specification is a simplified case of the general setting including all the possible scenarios related to the obligors' migrations (see the remark below). Namely, as it will appear evident soon, only a very limited subset of the possible $(7+1)^7$ scenarios is actually selected and the transition time (which in principle could occur anytime by the end of the time horizon) is fixed at $t = 0^+$.

   We can interpret this situation as if rating classes do not modify their features. What changes is the distribution of obligors across them, mimicking a sort of migration towards classes different from the original ones.[4] That means that the relative variation of PDs and LGDs is not fixed across classes and obligors belonging to those at the borders of the rating scale (either the best or the worst) cannot move beyond such limits.

Remark 5.1. ***Scenarios and migration matrices****. As anticipated in Section 2, the scenario setting describes the possible evolution of the rating properties of each obligor (see eq. (1)).*

Therefore, if properly chosen, scenarios are apt to define implicitly the transition probability of each obligor, belonging to a give initial rating class (one of those belonging to the tern of vectors (p^0, μ^0, σ^0)*), towards other rating classes. If we consider the rating specification introduced in the previous section, characterized by 7 rating classes ranging from* AAA *to* CCC *(according to typical scorings), at the level of single obligor, the migration can occur towards (7+1) rating classes, the last one being the default pseudo-rating class, denoted by D, whose value of* PD *equals 1 (this actually resembles the migration matrix associated to a single credit exposure).*

At the portfolio level, obligors can be grouped according to their rating features into rating classes. In this framework, the theoretical model is suitable to describe the joint evolution of such classes towards others, implicitly defining the corresponding joint transition probabilities. For instance, in the current setting, the total number of possible scenarios is equal to $(7+1)^7 \sim 2$ *millions. Such a high number is practically unfeasible to handle; however, it can in principle be reduced by limiting to special cases of migrations. An example will be shown below.*

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[4]In the simulation we present we do not consider the possibility to migrate towards a *default* class from any of the other rating classes.

As for the first type of implementation, we consider the following examples:

$$1A: \quad (p,\mu) \quad = \quad \begin{cases} (p^0,\mu^0) & \text{with } \lambda_1 = 80\% \\ (p^0(1-50\%),\mu^0(1-25\%)) & \text{with } \lambda_2 = 5\% \\ (p^0(1+50\%),\mu^0(1+25\%)) & \text{with } \lambda_3 = 15\% \end{cases}$$

$$1B: \quad (p,\mu) \quad = \quad \begin{cases} (p^0,\mu^0) & \text{with } \lambda_1 = 80\% \\ (p^0(1+20\%),\mu^0(1+10\%)) & \text{with } \lambda_2 = 10\% \\ (p^0(1+50\%),\mu^0(1+25\%)) & \text{with } \lambda_3 = 10\% \end{cases}$$

$$1C: \quad (p,\mu) \quad = \quad \begin{cases} (p^0,\mu^0) & \text{with } \lambda_1 = 90\% \\ (p^0(1+70\%),\mu^0(1+35\%)) & \text{with } \lambda_2 = 10\% \end{cases}$$

$$1D: \quad (p,\mu) \quad = \quad \begin{cases} (p^0,\mu^0) & \text{with } \lambda_1 = 80\% \\ (2p^0,\mu^0(1+50\%)) & \text{with } \lambda_2 = 15\% \\ (3p^0,2\mu^0) & \text{with } \lambda_3 = 5\% \end{cases}$$

In order to illustrate the second type of implementation, it proves useful to introduce for notational reasons operators which define the transition from one rating class to another. Let us define $\widehat{P}_k$, $k = \pm 1 \ldots \pm 7$ such that:

$$\widehat{P}_k p_j^0 = p_{j+k}^0 \qquad \text{and} \qquad \widehat{P}_k \mu_j^0 = \mu_{j+k}^0$$

where j indicates the rating class ($j = 1$ corresponds to AAA, $j = 2$ to AA and so on) and we adopt the convention on the signs such that $-k$ stands for an improvement of k rating classes and $+k$ for a deterioration of k classes. Border values on the rating scale deserve a special treatment. For example, we set:

$$\widehat{P}_{-1} p_1^0 = p_1^0 \qquad \text{and} \qquad \widehat{P}_{+1} p_7^0 = p_7^0$$

and so on. We then consider the following cases. At $t = 0^+$:

$$2A: \quad (p,\mu) \quad = \quad \begin{cases} (p^0,\mu^0) & \text{with } \lambda_1 = 80\% \\ \widehat{P}_{-1}(p^0,\mu^0) & \text{with } \lambda_2 = 10\% \\ \widehat{P}_{+1}(p^0,\mu^0) & \text{with } \lambda_3 = 10\% \end{cases}$$

$$2B: \quad (p,\mu) \quad = \quad \begin{cases} (p^0,\mu^0) & \text{with } \lambda_1 = 95\% \\ \widehat{P}_{+2}(p^0,\mu^0) & \text{with } \lambda_3 = 5\% \end{cases}$$

$$2C: \quad (p,\mu) \quad = \quad \begin{cases} (p^0,\mu^0) & \text{with } \lambda_1 = 80\% \\ \widehat{P}_{+1}(p^0,\mu^0) & \text{with } \lambda_2 = 10\% \\ \widehat{P}_{+2}(p^0,\mu^0) & \text{with } \lambda_3 = 10\% \end{cases}$$

5.3 Scenario analysis

Given the setup of the previous sections, we present the results of our scenario analysis, based on scenarios of type 1 and 2, introduced in the last paragraph of Section 5.2.

Scenario	M	$t_{99.9\%}(L)$	$t_{99.9\%}(\overline{L})$	Δ_{YC}	Δ_C	Δ_{GA}	Δ_∞
Sgl		0.0553	0.0357	0.0196	0.0071	0.0120	0.0076
1A		0.0635	0.0378	0.0256	0.0076	0.0125	0.0132
1B	200	0.0618	0.0385	0.0233	0.0076	0.0125	0.0108
1C		0.0647	0.0386	0.0261	0.0076	0.0126	0.0135
1D		0.1086	0.0468	0.0618	0.0093	0.0141	0.0477
Sgl		0.0526	0.0413	0.0113	0.0066	0.0042	0.0070
1A		0.0603	0.0436	0.0167	0.0070	0.0044	0.0123
1B	500	0.0588	0.0444	0.0144	0.0070	0.0044	0.0101
1C		0.0614	0.0445	0.0169	0.0071	0.0044	0.0125
1D		0.1023	0.0535	0.0488	0.0085	0.0050	0.0438
Sgl		0.0516	0.0428	0.0088	0.0063	0.0021	0.0067
1A		0.0592	0.0453	0.0140	0.0067	0.0021	0.0119
1B	1000	0.0578	0.0460	0.0118	0.0067	0.0021	0.0097
1C		0.0604	0.0461	0.0142	0.0067	0.0021	0.0121
1D		0.1002	0.0570	0.0449	0.0081	0.0024	0.0425
Sgl		0.0513	0.0443	0.0071	0.0071	0.0004	0.0067
1A		0.0589	0.0467	0.0122	0.0066	0.0004	0.0118
1B	5000	0.0575	0.0475	0.0100	0.0066	0.0004	0.0096
1C		0.0600	0.0476	0.0124	0.0067	0.0004	0.0120
1D		0.0995	0.0570	0.0425	0.0080	0.0005	0.0421

Table 2: Results obtained for an average quality portfolio, characterized by 7 rating classes, 11 industry-geographic areas and contagion factors, at the level of confidence $q = 99.9\%$.

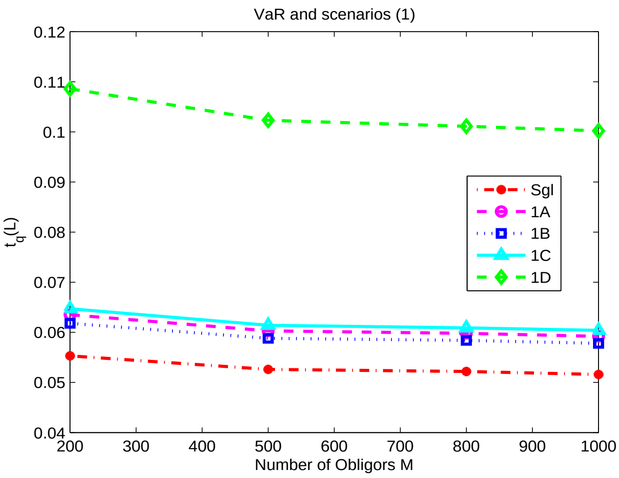

Figure 1: Approximated VaR, $t_q(L)$, at level of confidence $q = 99.9\%$ vs number of obligors M, for different scenarios of type 1.

VaR decomposition

We start from scenarios of type 1. Table 2 summarizes the main results, including both the final calculation of the approximated VaR and of its constituent components. From the data collected in Table 2, we can visualize in Fig. 1 the behavior of the approximated VaR, $t_q(L)$ $(q = 99.9\%)$, versus the number of obligors M, for different scenarios. The red line represents the single scenario situation and given the cases analyzed it is associated to the lowest VaR. All other cases, except from the most conservative 1D, deviate from the (Sgl) single scenario's VaR, of about 10÷15%. Case 1D, which entails duplication and triplication of the PDs values, shows a variation of VaR which is roughly twice as much as the value in the single scenario case.

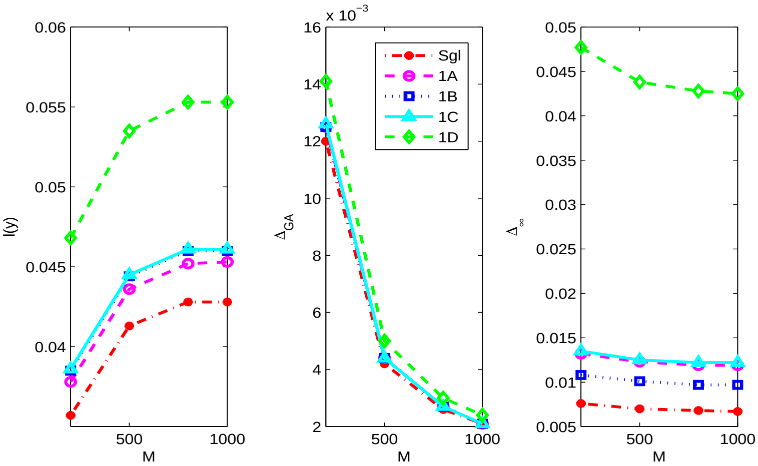

Figure 2: Decomposition of the second order VaR, $t_q(L)$ into its main components: zeroth order term $t_q(\overline{L}) \equiv l(y)$, with $y = N^{-1}(1-q)$ and $q = 99,9\%$, granularity adjustment Δ_{GA} and multi-factor correction Δ_∞.

Fig. 2 shows the decomposition of the approximated VaR in terms of its main contributions, i.e. the zeroth order term $t_q(\overline{L})$ and the corrections due to granularity in the exposures and the multi-factor setup. First, we comment on trends which are common to all scenarios, then we analyze the behavior of different scenarios:

- while the asymptotic VaR is an increasing function of the number of obligors, the resulting downward sloping curves in Fig. 1 are due to the effects of second order corrections. As the number of obligors increases, Δ_∞ tends towards a steady value, while the granularity adjustment becomes progressively negligible (see Table 2).
- analyzing the influence of different scenario choices, we notice that the major role is played by the zeroth order term $t_q(\overline{L})$ and the second order correction Δ_∞, the granularity adjustment being only mildly affected by it.

We have performed an analogous analysis on the second type of scenarios, which involve transitions across rating classes. The results are displayed in Fig. 3. As expected, though border classes are only partially affected by the presence of different scenarios, on average the values of the PDs and LGDs vary more drastically in this framework, leading to higher values of the approximated VaR. In the cases considered, the single scenario value at risk appears to be doubled or even tripled in the most conservative case $2C$.

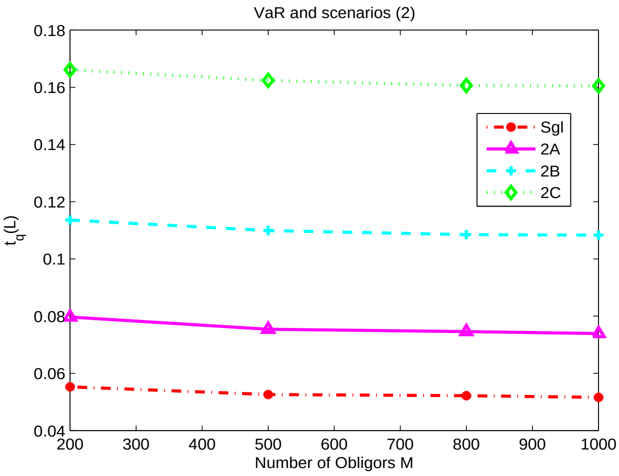

Figure 3: Approximated VaR, $t_q(L)$, at level of confidence $q = 99.9\%$ vs number of obligors M, for different scenarios of type 2.

Total approximated VaR vs weighted sum of VaRs

We now compare the results obtained in the theoretical framework, through eq.s (17), (18), (28) and (29), with the sum of single scenario VaRs, namely $[t_q(L)]_\varphi$, each weighted by the appropriate probability λ_φ:

$$\text{weighted sum} = \sum_{\varphi=1}^{S} \lambda_\varphi \, [t_q(L)]_\varphi \, .$$

Scenario	$t_{99.9\%}(L)$	weighted sum	$\Delta_\%$
1A	0.0603	0.0553	8.33%
1B	0.0588	0.0562	4.52%
1C	0.0614	0.0563	8.41%
1D	0.1023	0.0664	35.10%
2A	0.0754	0.0553	26.65%
2B	0.1099	0.0598	45.58%
2C	0.1624	0.0728	55.17%

Table 3: Comparison between the approximated VaR, $t_{99.9\%}(L)$, and the weighted sum of individual VaRs corresponding to different scenarios (of type 1 and 2) for $M = 500$.

Table 3 shows the outcomes for different scenarios of type 1 and 2. The last column reports the values of the percentage difference obtained as

$$\Delta_\% = \frac{t_{99.9\%}(L) - \text{weighted sum}}{t_{99.9\%}(L)} \cdot 100\% \, .$$

Such a quantity is of the order of $5 \div 10\%$ for scenarios which are not too distant from the single scenario case (e.g. $1A$, $1B$ and $1C$), but becomes extremely relevant for conservative ones, the discrepancy being more pronounced for scenarios of type 2, where $\Delta_{\%}$ ranges from about 26% to 55%. Results can be visualized in Fig. 4. Therefore, the weighted sum over different scenarios may significantly underestimate the true (and also the second order approximated) value at risk.

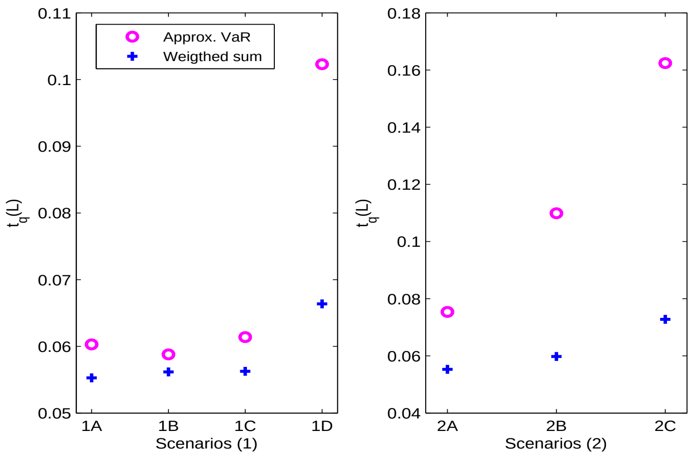

Figure 4: Comparison between the approximated VaR, $t_q(L)$ with $q = 99.9\%$, and the weighted sum of individual VaRs corresponding to different scenarios (of type 1 and 2) for $M = 500$.

VaR vs probability weights

We here analyze the behavior of the approximated value at risk as a function of the probability weights λ_φ. In order to keep things simple, we consider cases based on two scenarios only:

$$1E: \quad (p,\mu) \quad = \quad \begin{cases} (p^0,\mu^0) & \text{with } \lambda_1 = (1-\lambda) \\ (p^0(1+50\%),\mu^0(1+25\%)) & \text{with } \lambda_2 = \lambda \end{cases}$$

$$2D: \quad (p,\mu) \quad = \quad \begin{cases} (p^0,\mu^0) & \text{with } \lambda_1 = (1-\lambda) \\ \widehat{P}_{+1}(p^0,\mu^0) & \text{with } \lambda_2 = \lambda \end{cases}$$

Letting $\lambda_2 \equiv \lambda$ vary from zero to 80% we collect the values of $t_{99.9\%}(L)$ in Table 4.

			$t_{99.9\%}(L)$		
Scenario	$\lambda_2 = 0\%$	$\lambda_2 = 20\%$	$\lambda_2 = 40\%$	$\lambda_2 = 60\%$	$\lambda_2 = 80\%$
1E	0.0526	0.0621	0.0690	0.0738	0.0768
2D	0.0526	0.0880	0.1077	0.1162	0.1164

Table 4: Approximated VaR, $t_{99.9\%}(L)$, for two scenarios, as a function of the second scenario weight λ_2 for $M = 500$.

Plotting the results in Fig. 5 for cases 1E (green line) and $2D$ (red line) against the probability weight λ_2, we notice that the relationship between the value at risk and the probability weight of the second scenario is not linear. All curves are characterized by a common intercept, viz the value at risk of the single scenario case $t_{99.9\%}(L) = 0.0526$ and $\lambda_2 = 0$. Upon increasing the value of λ_2 curves associated to less dramatic scenarios appear flatter, in contrast to those describing conservative ones, which appear immediately steeper and with a higher curvature.

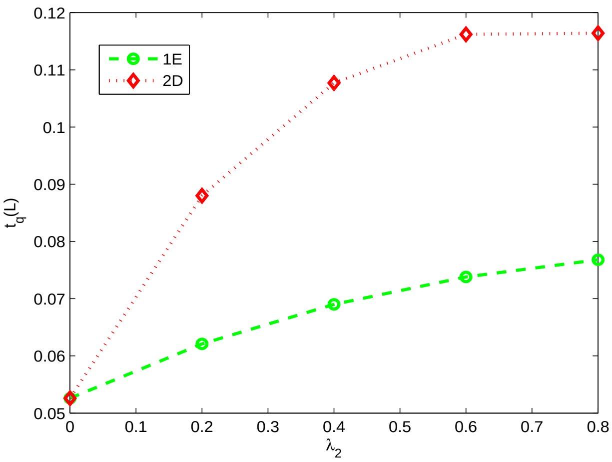

Figure 5: Approximated VaR, $t_{99.9\%}(L)$, for two scenarios, as a function of the second scenario weight λ_2 for $M = 500$.

5.4 Sector Concentration Analysis

We now investigate the role of sector concentration, focusing on its impact (if any) on the scenario analysis. The study of the previous paragraph has been conducted assuming a portfolio of loans homogeneously distributed across industry-geographic areas. We now consider a portfolio concentrated mainly in two sectors. The comparison between the approximated VaR obtained in this case and the one corresponding to a uniformly distributed portfolio is shown in Table 5 and Fig. 6 for $M = 500$ obligors, and scenarios of the first type.

Fig. 6 displays on the x-axis different scenarios, ranging from Sgl to 1C. The y-axis reports the values of the approximated VaR, $t_q(L)$. Interpolating the resulting points for more clarity, it turns out that the corresponding curves run almost parallel. Therefore, the effect of sector concentration just produces a constant shift in $t_q(L)$, resulting in higher values for more concentrated portfolios, as expected.

	Homogeneous ptf. $t_{99.9\%}(L)$	Concentrated ptf. $t_{99.9\%}(L)$
Sgl	0.0526	0.0638
1A	0.0603	0.0719
1B	0.0588	0.0706
1C	0.0614	0.0731

Table 5: Approximated VaR, $t_{99.9\%}(L)$, for different scenarios of type 1, corresponding to a portfolio of loans homogeneously distributed across sectors *and* a portfolio concentrated in two sectors ($M = 500$).

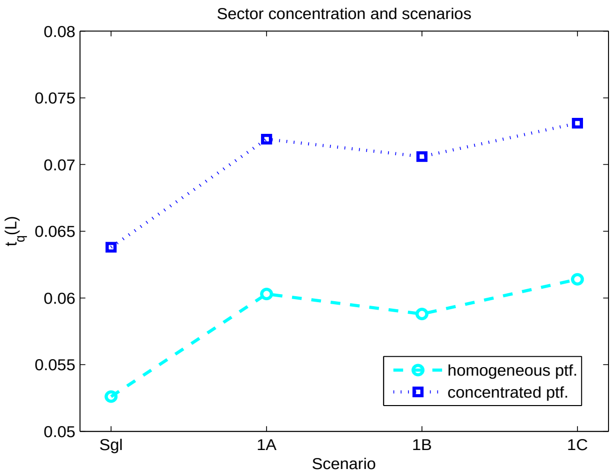

Figure 6: Approximated VaR, $t_q(L)$ with $q = 99.9\%$, for different scenarios of type 1, corresponding to a portfolio of loans homogeneously distributed across sectors *and* a portfolio concentrated in two sectors ($M = 500$).

5.5 Contagion Analysis

We conclude the numerical analysis by studying the effects of contagion. In particular, we focus on the role played by the number of infecting segments acting on each obligor. In the previous analysis, to simplify things, we opted for just two contagion sectors. Here we compare the previous results with those obtained by incrementing the number of contagion sectors. Explicitly, we consider five and eleven sectors, in addition to the original two, assuming uniform participation coefficients, δ^C_{ik}, for each obligor (see Appendix A2). To keep things general we choose two scenarios, Sgl and 1B, and number of obligors $M = 500$ and 1000.

The complete results are collected in Table 6 while Fig. 7 highlights the main aspects. The picture shows the behavior of $t_q(L)$ ($q = 99.9\%$) and of the total contagion correction Δ_C versus the number of contagion sectors (we connected the three points with lines). A peak occurs in correspondence of the intermediate number of sectors (in this case five). This is consistent with intuition. Starting from a low

Ctg. sectors	Scenario	M	$t_{99.9\%}(L)$	$t_{99.9\%}(\overline{L})$	Δ_{YC}	Δ_C	Δ_{GA}	Δ_∞
2			0.0526	0.0413	0.0113	0.0066	0.0042	0.0070
5	Sgl	500	0.0580	0.0413	0.0168	0.0121	0.0042	0.0125
11			0.0484	0.0413	0.0071	0.0024	0.0042	0.0029
2			0.0516	0.0428	0.0088	0.0063	0.0021	0.0067
5	Sgl	1000	0.0569	0.0428	0.0140	0.0115	0.0021	0.0120
11			0.0477	0.0428	0.0049	0.0024	0.0021	0.0028
2			0.0588	0.0444	0.0144	0.0070	0.0044	0.0101
5	1B	500	0.0646	0.0444	0.0202	0.0128	0.0044	0.0158
11			0.0544	0.0444	0.0100	0.0026	0.0044	0.0056
2			0.0578	0.0460	0.0118	0.0067	0.0021	0.0097
5	1B	1000	0.0634	0.0460	0.0174	0.0123	0.0021	0.0152
11			0.0537	0.0460	0.0076	0.0025	0.0021	0.0055

Table 6: Approximated VaR, $t_{99.9\%}(L)$, and its components, for different choices of the number of contagion sectors (2,5 or 11), number of obligors M and scenarios Sgl (single-scenario) and 1B.

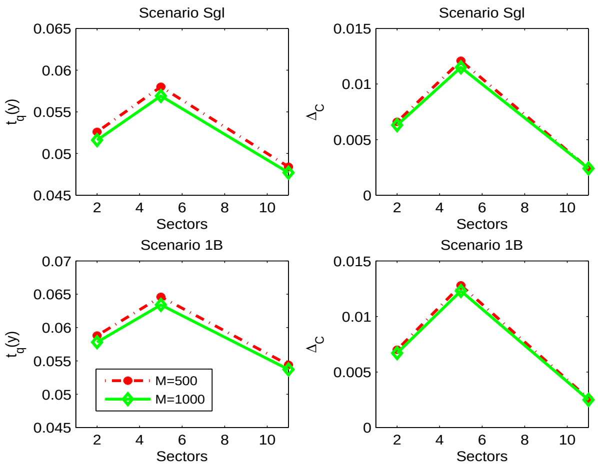

Figure 7: Approximated VaR, $t_{99.9\%}(L)$, and contagion correction Δ_C, for different choices of the number of contagion sectors (2,5 or 11), number of obligors M and scenarios Sgl (single-scenario) and 1B.

number of sectors, when increasing it, the effects of contagion become more relevant, till the moment in which sector diversification starts to predominate, thus leading to a reduction of the contagion adjustment and consequently of the total VaR. This effect is particularly evident here, given our choice involving uniformly distributed participation weights.

6 Conclusions

This paper deals with the *analytical* computation of the value at risk for a portfolio of loans in the presence of credit risk. This approach represents a valid alternative to otherwise computationally heavy Monte Carlo simulations.

Besides delving into the well known issues regarding concentration risk (*single-name*, *sector concentration* and *contagion*) already exposed in [3], we propose a new perspective which allows to model in a more flexible and *non static* way the rating properties (PDs and LGDs) of obligors, and the link between the level of default probabilities and the losses given default. This is achieved through the introduction of different possible scenarios, each characterized by distinctive rating features and weight. Each obligor, initially assigned to a rating class, at a successive instant of time can change his rating properties according to a given set of scenarios. As a byproduct, PDs and LGDs which are assumed independent in each single scenario, turn out to be *implicitly* correlated in the wider picture. This scenario setup is very rich and flexible, allowing to choose among several structures, amenable of different interpretations.

In the paper we have shown three of them, including a simplified case which mimics migrations of obligors between rating classes, and the results indicate a noteworthy increment of the value at risk of a credit portfolio.

Appendix

A Parameters' description

In this Appendix we provide a detailed description of the parameters which enter the model as inputs. In particular, we highlight the relationship between such quantities and the data available to banks. In the first paragraph we focus on the inputs necessary to the multi-factor setting, in the last one we turn our attention to the parameters defining the contagion specification of the theoretical model.

A.1 Multi-factor parameters

We outline the main steps in order to obtain the standardized asset returns $X_{i\varphi}$ (eq. (5)), focusing on the multi-factor part only. A thorough analysis about how to choose industry-geographic sectors from observable data and extract the necessary information from them (specifically about their distributional properties) can be found in [16]. Here, we assume this relevant information is already available and we focus on how to characterize individual obligors' asset returns through an appropriate mapping to sectors.

1. Consider a set of N industry-geographic sectors I_k, $k = 1, \dots, N$, each of which is assumed to be distributed according to a normal distribution. In matrix notation, the vector of sectors is $I \sim \mathcal{N}(\overline{I}, \Sigma)$, with $\overline{I}$ the vector of mean values and Σ the $N \times N$ variance-covariance matrix, encoding correlations among sectors.

2. We assign weights to each obligor i taking into account:
   - its sensitivity to firm-specific, idiosyncratic risk, expressed through the factor loading $r_{i\varphi}$, with $0 \le r_{i\varphi} \le 1$. For implementation purposes, we choose to express such a parameter as a function of the probability of default[5]. In this way, factor loadings $\{r_{i\varphi}\}$ can be grouped into buckets based on the rating properties of the corresponding obligors.
   - its participation in industry and geographical sectors. This feature is captured by the coefficients δ_{ik} which appear in the decomposition of the systematic composite factor $\widehat{Y}_i$ in terms of the sectors I_k:

   $$\widehat{Y}_i = \sum_{k=1}^{N} \delta_{ik} \, I_k \,,$$

   (the hat symbol indicates the fact that the composite factor is not standardized yet). The industry participation coefficient δ_{ik} represents the percentage of the business activity of obligor i into the kth sector.

   Normally distributed (but not yet standardized) asset returns assume the form

   $$\widehat{X}_{i\varphi} = r_{i\varphi} \sum_{k=1}^{N} \delta_{ik} \, I_k + \sqrt{1 - r_{i\varphi}^2} \, \xi_i \,.$$

3. We are interested in the standardized asset returns $X_{i\varphi}$. Given

   $$\widehat{\sigma}_i \equiv \sqrt{\mathrm{var}(\widehat{Y}_i)} = \sqrt{\sum_{k,l=1}^{N} \delta_{ik}\delta_{il} \, \mathrm{Cov}(I_k, I_l)} \,,$$

   the standardized composite factor reads $Y_i = (\widehat{Y}_i - \sum_{k=1}^{N} \delta_{ik}\overline{I}_k)/\widehat{\sigma}_i$, where $\overline{I}_k$ stands for the mean value of the industry-geographic sector index I_k. Therefore, eq. (2) is recovered.

4. The final step consists in expressing Y_i as a linear combination of *independent* standard normally distributed factors $Z_k \sim \mathcal{N}(0, 1)$:

   $$Y_i = \sum_{k=1}^{N} \alpha_{ik} \, Z_k \,,$$

   where the coefficients α_{ik} must be matched to the parameters δ_{ik}, $\rho_{k,l}$ and $\widehat{\sigma}_i$. This task can be achieved through the Cholesky decomposition of the vector of sectors $I - \overline{I} \sim \mathcal{N}(0, \Sigma)$, in terms of the matrix A such that $I - \overline{I} = AZ$ and

[5] Such a choice mimics the definition of systematic factor loadings proposed by the Basel Committee on Bank Supervision [20].

$AA^T = \Sigma$ (e.g. see Glasserman for the Cholesky algorithm [11]). It follows

$$\begin{aligned} I_1 - \overline{I}_1 &= A_{11} Z_1 \,, \\ I_2 - \overline{I}_2 &= A_{21} Z_1 + A_{22} Z_2 \,, \\ &\vdots \\ I_N - \overline{I}_N &= A_{N1} Z_1 + \ldots + A_{NN} Z_N \,. \end{aligned} \tag{30}$$

After standardization, we obtain the final result

$$\begin{aligned} \alpha_{i1} &= \frac{\delta_{i1}}{\widehat{\sigma}_i} A_{11} + \frac{\delta_{i2}}{\widehat{\sigma}_i} A_{21} + \ldots + \frac{\delta_{iN}}{\widehat{\sigma}_i} A_{N1} \,, \\ \alpha_{i2} &= \frac{\delta_{i2}}{\widehat{\sigma}_i} A_{22} + \ldots + \frac{\delta_{iN}}{\widehat{\sigma}_i} A_{N2} \,, \\ &\vdots \\ \alpha_{iN} &= \frac{\delta_{iN}}{\widehat{\sigma}_i} A_{NN} \,. \end{aligned} \tag{31}$$

By construction, $\sum_{k=1}^{N} \alpha_{ik}^2 = 1$ so that Y_i has unit variance.

A.2 Contagion parameters

The parameters to be estimated from market data are the factor loadings $\{g_i\}$ and the coefficients $\{\gamma_{ik}\}$ which appear in the expansion of the composite contagion factor Γ_i in terms of the latent variables C_k. The idea we propose in our model specification, in order to choose such parameters, is to rely on the information encoded into the revenues generated by single obligors.

We assume that data about the revenues of each obligor, R_i, are available. In particular, we assume it is possible to quantify the amount of revenues earned from transactions with the infecting segment of each sector. Let us call this quantity R_{ik}^I, where $k = 1, \ldots, N$ specifies the sector and $i = 1, \ldots, M$ the single obligor. The participation weight δ_{ik}^C, which gives the dependence of obligor i on the infecting segment of sector k, can therefore be expressed as:

$$\delta_{ik}^C = \frac{R_{ik}^I}{R_i^I} \,,$$

Realizing that the contagion composite factor (not yet standardized) can be written as

$$\widehat{\Gamma}_i = \sum_{k=1}^{N} \delta_{ik}^C \, I_k^I$$

where, with obvious notation, I_k^I represents the infecting segment of sector k, and applying the same Cholesky decomposition derived in the multi-factor case, it is possible to obtain the sought after contagion coefficients $\{\gamma_{ik}\}$.

B Effective factor loadings $\{a_{i\varphi}\}$

In this section we sketch a derivation of the coefficients $\{a_{i\varphi}\}$ in the most general case. Simplifications may apply to properly chosen portfolios. Let us start recalling that, for any scenario φ, $a_{i\varphi}$ encodes the correlation properties of the asset return $X_{i\varphi}$ with the effective systematic risk factor $\overline{Y}$, i.e.

$$a_{i\varphi} = r_{i\varphi}\,\mathrm{corr}(Y_i, \overline{Y}) = r_{i\varphi} \sum_{k=1}^{N} \alpha_{ik} b_k \,.$$

While the coefficients α_{ik} are given as an input, the choice of $\{b_k\}$ is not unique. This follows from the observation that the set $\{b_k\}$ specifies the zeroth-order term in the Taylor expansion, $t_q(\overline{L})$, whose only requirements are to be analytically tractable and *close enough* to $t_q(L)$.

The idea is to find the optimal single effective risk factor $\overline{Y}$, which minimizes the difference between $t_q(L)$ and $t_q(\overline{L})$. A detailed explanation of the necessary optimization procedures in the single scenario case is presented in [21]. The result can be summarized through the following steps:

- define the variable

$$c_i = w_i \sum_{\varphi=1}^{S} \mu_{i\varphi}\, N\left[\frac{N^{-1}(p_{i\varphi}) + r_{i\varphi}\, N^{-1}(q)}{\sqrt{1 - r_{i\varphi}^2}}\right] ,$$

  and express the coefficient b_k in terms of c_i

$$b_k = \sum_{i=1}^{M} \frac{c_i}{\Lambda}\, \alpha_{ik} \,,$$

  where Λ is a Lagrange multiplier to be determined.

- Impose the requirement of unit variance for $\overline{Y}$, i.e. $\sum_{k=1}^{N} b_k^2 = 1$ and derive the value of Λ.

- Plug Λ and c_i into the definition of b_k.

C Formulae

C1 Analytical derivation of $t_q(\overline{L})$, eq. (16)

We want to prove that

$$t_q(\overline{L}) = l(N^{-1}(1-q))\,,$$

where $l(\overline{Y})$ is defined by eq.s (14) and (15).

We start from the definition of quantile at level q, i.e.:

$$t_q(\overline{L}) : \quad \mathbb{P}(\overline{L} \le t_q(\overline{L})) = q\,.$$

Therefore, we look for a variable ℓ satisfying

$$\mathbb{P}(\overline{L} \leq \ell) = q\,. \tag{32}$$

Applying the definition of $\overline{L}$

$$\overline{L} \equiv l(\overline{Y}) = \sum_{i=1}^{M} w_i \sum_{\varphi=1}^{S} \lambda_{i\varphi}\mu_{i\varphi}\,\hat{p}_{i\varphi}(\overline{Y})$$

where

$$\hat{p}_{i\varphi}(y) = N\left[\frac{N^{-1}(p_{i\varphi}) - a_{i\varphi}y}{\sqrt{1-a_{i\varphi}^2}}\right],$$

we can rewrite eq. (32) as follows

$$\mathbb{P}\left(\sum_{i=1}^{M} w_i \sum_{\varphi=1}^{S} \lambda_{i\varphi}\mu_{i\varphi}\,\hat{p}_{i\varphi}(\overline{Y}) \leq \ell\right) = q\,. \tag{33}$$

$\overline{L}$ is a sum of deterministic and strictly decreasing functions $\hat{p}_{i\varphi}$ of $\overline{Y}$. Moreover $\overline{L}$ is bounded, assuming values in the interval $[0, \overline{\ell}]$, where

$$\overline{\ell} = \sum_{i=1}^{M} w_i \sum_{\varphi=1}^{S} \lambda_{i\varphi}\mu_{i\varphi}\,.$$

Therefore $\overline{L} = l(\overline{Y})$ can be inverted and

$$\exists! \quad y_\ell\ : \qquad \sum_{i=1}^{M} w_i \sum_{\varphi=1}^{S} \lambda_{i\varphi}\mu_{i\varphi}\,\hat{p}_{i\varphi}(y_\ell) = \ell \in [0, \overline{\ell}]\,.$$

Expressing eq. (33) in terms of the variable $\overline{Y}$, we get

$$q = \mathbb{P}(\overline{Y} > y_\ell) = 1 - \mathbb{P}(\overline{Y} \leq y_\ell)\,,$$

so that ℓ is such that:

$$\mathbb{P}(\overline{Y} \leq y_\ell) = 1 - q\,.$$

Since $\overline{Y} \sim \mathcal{N}(0,1)$, the corresponding quantile y_ℓ turns out to be

$$N(y_\ell) = 1 - q \;\Rightarrow\; y_\ell = N^{-1}(1-q)\,.$$

In terms of the original variable $\overline{L}$, we get the sought after expression of the quantile:

$$t_q(\overline{L}) \equiv \ell = \sum_{i=1}^{M} w_i \sum_{\varphi=1}^{S} \lambda_{i\varphi}\mu_{i\varphi}\,\hat{p}_{i\varphi}(N^{-1}(1-q)) = l(N^{-1}(1-q))\,.$$

C2 Derivatives of $l(y)$ and $\nu(y)$, eq. (18)

$$l'(y) = \sum_{i=1}^{M} w_i \sum_{\varphi=1}^{S} \lambda_{i\varphi}\mu_{i\varphi}\,\hat{p}'_{i\varphi}(y)\,,$$

$$l''(y) = \sum_{i=1}^{M} w_i \sum_{\varphi=1}^{S} \lambda_{i\varphi}\mu_{i\varphi}\,\hat{p}''_{i\varphi}(y)\,,$$

$$\hat{p}'_{i\varphi}(y) = -\frac{a_{i\varphi}}{\sqrt{1-a_{i\varphi}^2}}\; n\left[\frac{N^{-1}(p_{i\varphi}) - a_{i\varphi}y}{\sqrt{1-a_{i\varphi}^2}}\right],$$

$$\hat{p}''_{i\varphi}(y) = -\frac{a_{i\varphi}^2}{1-a_{i\varphi}^2}\,\frac{N^{-1}(p_{i\varphi}) - a_{i\varphi}y}{\sqrt{1-a_{i\varphi}^2}}\; n\left[\frac{N^{-1}(p_{i\varphi}) - a_{i\varphi}y}{\sqrt{1-a_{i\varphi}^2}}\right],$$

$$\begin{aligned}\nu'_\infty(y) &= 2\sum_{i,j=1}^{M} w_i w_j \sum_{\varphi,\psi=1}^{S} \mu_{i\varphi}\mu_{j\psi}\;\lambda_{i\varphi,j\psi}\;\hat{p}'_{i\varphi}(y)\;N\left(\frac{N^{-1}[\hat{p}_{j\psi}(y)] - \rho^{YC}_{i\varphi,j\psi}N^{-1}[\hat{p}_{i\varphi}(y)]}{\sqrt{1-(\rho^{YC}_{i\varphi,j\psi})^2}}\right) + \\ &- 2\sum_{i,j=1}^{M} w_i w_j \sum_{\varphi,\psi=1}^{S} \mu_{i\varphi}\mu_{j\psi}\lambda_{i\varphi}\lambda_{j\psi}\,\hat{p}'_{i\varphi}(y)\hat{p}_{j\psi}(y)\,,\end{aligned}$$

$$\begin{aligned}\nu'_{GA}(y) &= \sum_{i=1}^{M} w_i^2 \sum_{\varphi=1}^{S} \lambda_{i\varphi}(\mu_{i\varphi}^2 + \sigma_{i\varphi}^2)\hat{p}'_{i\varphi}(y) + \\ &- 2\sum_{i=1}^{M} w_i^2 \sum_{\varphi,\psi=1}^{S} \mu_{i\varphi}\mu_{i\psi}\lambda_{i\varphi,i\psi}\;\hat{p}'_{i\varphi}(y)N\left(\frac{N^{-1}[\hat{p}_{i\psi}(y)] - \rho^{YC}_{i\varphi,i\psi}N^{-1}[\hat{p}_{i\varphi}(y)]}{\sqrt{1-(\rho^{YC}_{i\varphi,i\psi})^2}}\right).\end{aligned}$$

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