Research Paper Series

Modelling of Libor-Ois Basis

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Executive Summary

The authors introduce a set of models that explain the market phenomenology of Libor forward fixings implied in swap prices. The models are all based on the idea that the Libor fixings refer to a panel of primary banks whose composition may change over time. This effect is crucial to obtain the observed humped forward fixing curves, that could not be otherwise retrieved by a simple credit default model or by a forward interest rate analogy. The models differ only in the assumptions on how the panel composition will change in the future.
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This article was written in collaboration with Matteo Camelia and Andrea Cova, whom at the time were working for Iason Consulting
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Modelling of Libor-Ois Basis

Matteo Camelia  Antonio Castagna  Andrea Cova

Since the financial crisis started in 2008 the Libor-Ois basis have been no more negligible: this implies a major change in the evaluation of interest rate derivatives and as a consequence the single curve interest rate models have become obsolete.

Ois rates can be approximately considered default risk-free due to the fact that they are derived from overnight deposit rates: therefore they embed the risk of default overnight even when they are the reference rate for longer maturities, e.g.: an Ois swap expiring in 10 years.

Libor spot and forward rates embed the risk that the borrower (a major bank belonging to the relevant Libor panel, depending on the currency the debt is denominated in) may go bankrupt before the expiry of the deposit. As such, a Libor rate for, say, a 6-month deposit, include a spread over the 6-month Ois rate to remunerate the lender for the risk of the borrower’s default over next 6 months. The Ois-Libor basis is typically increasing with maturities (from the O/N to 1 year) for spot Libor rates: one would expect also a similar behaviour for forward Libor rates, quoted as FRAs up to 24 months and implied in swap rates for longer maturities, but this is not what market rates exhibit.

It is now well known that the forward basis curves, for all tenors, show a “humped” shape: this phenomenon has been documented by some authors (see, for example, Morini [8] and Ametrano and Bianchetti [1], figure 35). They find that the basis curves are initially increasing until a certain future time, and then they start decreasing monotonically onwards, until an asymptotic value of a few basis points is reached. A confirmation of the persistence of this feature, even in a financial environment with lower rates and Libor-Ois basis than the one dealing in the period 2008-2010, is given in Figure 1, where we show the market rates in the EUR for swaps vs 3M, vs 6M, and Eonia on November 1st, 2014: from these quotes we show a very basic bootstrap of the basis Euribor 3M-Eonia and Euribor 6M-Eonia.

Although the “humped” pattern in both curves is easily recognisable, it is worth noting that the basis curves are very irregular even before the 10 year maturity, where the market is quite liquid and active. The weird slopes of the two curves become apparent around the 15 year maturity, where the market trades less frequently. Seemingly regular swap rate curves can generate greatly inconsistent shapes of the basis curves. A general model, based on grounds beyond the simple smoothing criteria, can be useful also to regularise, interpolate and extrapolate the Libor-Ois basis curves by fitting it to more liquid tenors.

Previous Basis Spread Modelling includes the works by Mercurio [6] and Moreni and Pallavicini [7]. In the first one, the author derives pricing formulae for linear and volatility derivatives, assuming stochastic dynamics for the single forward Libor-Ois basis spread, but no connection is established amongst spreads on the different relevant tenors (i.e.: 1, 3, 6, 12 months) and different future times. In the second work, the authors extend the HJM framework to account for a multi-curve environment: the model establishes a link amongst the forward Libor fixings at different future dates on the same tenor by means of their dependence on two common stochastic state variables, whose dynamics are capable to capture the nowadays typical humped term structure. The link between the Libor on different tenors is established via two deterministic scaling functions for the rate level and the volatility level. The framework is enough flexible to fit market prices, but no financial or economic rationale lies behind the type of functions and links chosen by the two authors.

In general, the approaches to model Libor-Ois spreads proposed so far by market practitioners and academicians aim just at matching market prices, usually with ad hoc assumptions, without trying to explain the evolution of the spreads by the credit factor that they represent. Although
FIGURE 1: Swap rate rates dealing in the market on November 1st, 2014 (left hand side) and implied Euribor-Eonia basis curves (right hand side). Source: swap indicative quotes provided by major brokers.

FIGURE 2: Forward credit spreads for deposits with respectively 1M, 3M, 6M and 1Y maturity, derived by assuming a stochastic default intensity that follows a CIR process of the type in equation (29), with parameters $\lambda_0 = 0.06\%, \kappa = 1.5, \theta = 1.0\%, \sigma = 5.0\%$.

these approaches can be fully justified on the grounds of their effectiveness to the purpose, nonetheless they rely on the existence of a liquid market where all types of main instruments (FRAs, swaps on Libor with all the tenors, Caps&Floors) are actively traded and quoted. When the market is not so liquid on some instruments (e.g.: swaps vs 1M Libor for maturities longer than 1 year), a general model can be used to evaluate them, even if it is calibrated on the traded liquid instruments. Clearly, this means that the model is able to deduce the Libor-Ois spreads for any future date and on any tenor, which implies it is based on the common risk factors driving all the spreads.

In what follows we introduce a unified set of models that are able to reproduce the “humped” shape of forward basis, yet that are capable to match the upward sloping basis curve for spot starting deposits. The models have some nice properties: i) they are based on the default risk generating the spreads, ii) they model simultaneously basis for all the (major) tenors (1, 3, 6 and 12 months), iii) they all rely on the factual assumption that the panel of banks, whom the Libor refers to, may change over time and that the any defaulting, or credit worsened, entity can be replaced within the panel itself.
1. The Main Idea Underpinning the Framework

The Libor rates can be thought to be made of two components: i) a risk-free part, generally considered to be equal to the Ois rate for the corresponding expiry, and ii) a credit spread that remunerates the lender for the credit risk it bears in lending money to a defaultable borrower. In our framework, the (Ois) risk-free rate is modelled as independent of credit spread; moreover, the credit spread is typically referred to in the market lore as the Libor-Ois basis. Besides, we will consider four major tenors: 1M, 3M, 6M and 1Y, used in most contracts; other spreads can be derived within the approach that we will outline, although they are less used as a reference index in interest rate derivatives. Additional factors, such as liquidity risk, are not directly considered in this set of models, although their inclusion is possible.

Classical credit spread models that consider a single counterparty, whose default is commanded by a stochastic default intensity, generate a set of monotonically increasing credit spread curves, starting from different initial values (spot spreads) for our four tenors and reaching a common asymptotic value. An example is shown in Figure 2.

Unfortunately this is not the type of term structures we observe in the market. The Libor-Ois (credit spread) basis does not simply represent the risk related to a single counterparty. Actually, the Libor rates are the interest rates, for the relevant maturity and currency, that a panel of major banks is expected to pay when borrowing money from a similar institutional counterparty. For reasoning’s sake, let us think of the Libor panel as identified by a single representative bank.

The representative bank of the Libor panel is an entity whose default risk may structurally change over time. By “structural change”, we do not simply mean the possibility that the default probability may stochastically evolve over time; we also mean that, since the representative bank is a sort of average synthesis of the default risks of the banks included in the panel, if the panel changes in its composition, then also the default risk of representative bank will change as well. This happens even if the probabilities of default of the banks currently included in the panel, and of the banks currently excluded, but which could potentially replace some of the former ones, are deterministic and known.1

To make things concrete, suppose the Libor panel is made of 10 banks all with a probability to go bankrupt over next year equal to 1%: the representative bank will trivially have a 1-year default probability of 1%. Assume now a bank replaces one of the current ones belonging to the panel, and let its default probability for 1 year be 0.8%. The representative bank should now have a 1-year default probability of 0.98%. Hence, its default risk has changed even if we did not assume any deterministic or stochastic evolution of the default probabilities of all the banks, either included or outside the Libor panel.

In the real world, when one of the banks currently belonging to the Libor panel experiences a worsening of its credit standing or even, in the extreme case, a default, then it is expected to be replaced by a new external bank, with a good credit standing that will likely improve the average credit quality of the panel. As a consequence, one would expect the credit spread of the representative bank to be lower.

The possibility that the panel changes its component banks is crucial to account for the humped shape of the forward Libor-Ois basis. Actually, restricting the observation to daily published Libor fixings, one immediately realise that Libor-Ois basis are increasing with the maturity of the deposits. This means that the market expects a rising probability of default over time. This is not very strange, as credit spread curves for single debtors, either corporates or banks, usually show the same upward slope. One would expect that the Libor-Ois basis for future dates (embedded in the forward rates applied to forward starting deposits) show an upward slope too; on the contrary, the prices of FRAs and of swaps quoted in the market imply a downward slope of the forward rates, after an initial increase up to the maturities of 3 - 5 years. Besides, even if market forward spreads are raising, they do not reflect the forward spreads implied in the Libor spot rates (see, for example, Mercurio [5]).

From this phenomena, it is possible to deduce the reasoning the lenders follow in setting spot and forward rates: if one has to lend today (spot) a given amount of cash to a bank of the panel,

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1We do not claim we are introducing some revolutionary idea here: we are simply trying to expose what is very likely the way market agents (traders) think when they need to make a price for a spot starting deposit or for a FRA. Similar explanations of the Libor panel composition, and change of it, have been proposed also in older works, such as in Morini [8].
she knows exactly which is the default risk she would bear. This risk is condensed in the Libor-Ois spreads for spot starting deposits, which are increasing in time, meaning that a higher probability of default for the representative bank is attached to longer maturities.

If the deposit is forward starting (as the one underlying FRAs), then the lender should account for the fact that on that future date the representative bank is no more the same as today, since some of the banks composing the panel may be replaced by new ones. The replacement can be due to the credit worsening or by the default of one or more banks; the new banks entering in the panel to replace the excluded ones will have very likely a better credit standing that would improve the average credit quality of the panel and hence of the representative bank, to which the forward (FRA) Libor rates refers to. For this reason, the forward Libor fixings implied in the FRAs’ and swaps’ market price are not increasing, as the spot spread curve would suggest, but decreasing, to take into account the general expected improvement of the credit quality of the panel over time, originated be the possible changes of the panel of banks.

To summarise, the Libor-Ois forward basis is actually a weighted average of forward spreads of the single members of the panel. If we assume that the market expects a likely future change of the composition of the panel (i.e.: some panel banks could be replaced by new banks) and / or a change in the credit worthiness of the current members, then we would not get monotonically increasing curves anymore. We will try to give a more visual representation of this concept

For simplicity, let us begin by assuming that market expects only improvements, that is: changes in panel composition and / or panel members’ credit standing that would result in a lower average credit spread. Given the current spot Libor-Ois basis curve, we would now have to include the future possibility of a transition to a substantially lower basis curve when computing expected future basis.

The resulting basis curve could then be seen as a gradual transition from a high Libor-Ois basis curve, denoted as $H$ and corresponding to the current panel, to a low Libor-Ois basis curve, denoted as $L$. Assume the market will begin to monitor the new lower credit spread curve from a certain future time onwards; therefore we can imagine the $L$ basis curve as starting with a certain time delay, as can be seen in Figure 3. At each time in the future, there will be a certain probability of a shift from the Libor-Ois basis curve $H$ to the curve $L$: at time 0 the expected Libor-Ois basis curve will resemble the one depicted in the figure.

The set of models we will present in this article, all share this basic idea; they differ only in the specific risk-neutral dynamics of the expected future panel changes. To make the models analytically tractable and usable in practice, we make the simplifying assumption that only two types of banks exist in the market: $H$ and $L$, with the former having a higher credit risk than the latter. These two types of banks have the same type of dynamics for their respective default intensities, although a

![FIGURE 3: A simplified explanatory graph depicting the gradual transition from a High forward Libor-Ois basis curve, to a Low forward Libor-Ois basis curve (starting delayed), due to a change of the representative bank.](image-url)
different set of parameters refers to the $H$ and the $L$ class.

To specify the risk-neutral dynamics commanding the expected gradual transition from the $H$ to the future $L$ basis curve, let us start assuming that the initial Libor panel is given in its composition of $H$ and $L$ banks. We consider three different assumptions for the transition dynamics, each one producing a different final Libor-Ois basis modelling:

1. The initial Libor panel is made of a single representative bank that may be totally replaced by a new, different, representative bank, according to a continuous-time Markov process with two states ($H$ and $L$) and a well defined instantaneous transition matrix.

2. The initial Libor panel is made of a given number of banks that differ in type ($H$ or $L$): the replacements occur at random discrete future times, driven by continuous-time Markov process. At every random replacement time, each of the banks in the panel can be replaced by another bank of any type, according to a transition matrix.

3. The Libor panel is a continuous weighted average of banks that differ in type ($H$ or $L$) and timing of entry in the panel: a deterministic continuous-time replacement process drives the gradual replacement dynamics towards a different panel composition.

All of these models assume that the replacement dynamics are independent from the single-counterparty default intensity dynamics and the Ois instantaneous rate dynamics.

We will structure the paper as follows:

1. Specify the assumptions for Ois instantaneous rate dynamics and derive the caplet and floorlet prices.

2. Specify the assumptions for single-counterparty default intensity dynamics and derive the results for classical credit spreads.
3. Specify separately for the three model variants, the transition dynamics assumptions and derive the respective results for expected credits spreads, spread probability density functions, Libor caplet prices and implied volatility smiles.

**Remark 1.1 (Modular Approach).** We wish to highlight that one of the features of our approach to basis modelling is modularity (see Figure 4). Actually, we separately specify:

1. the dynamics for the (Ois) risk-free rate;
2. the default intensity dynamics, having two different sets of parameters for H and L, but also sharing an identical type of dynamics for the two types of banks;
3. the panel reshuffling/transition dynamics, according to the chosen assumption amongst the three proposed above;

Moreover, we assume a mutual independence between all of these separate dynamics.

All this means that the user is then free to choose her own preferred dynamics for the Ois and the default intensities, even if we will adopt a CIR dynamics for all of them in what follows. Our choice should be regarded as taken just for explanatory purposes.

### 2. Libor-Ois Basis Modelling

Assume that, for a given reference currency, a group of (major) banks enter the Libor panel at time $t$. To simplify the analysis, we assume they are all equal to a representative bank that can go bankrupt with known default probabilities for any future date. The default probabilities can be considered as an average of the default probabilities of the single banks of the panel: one may think that if she lends money at time $t$ to one of these banks, she will bear an expected default risk equal to that referring to the representative bank.\(^2\)

Consider for the moment that the representative bank is exactly similar to a specific institution with its own default risk; alternatively said, lending spot, or forward at a future date, an amount of money to the representative bank is no different than lending to a specific bank operating in the market that wishes to borrow (we will relax this assumption soon). The (risk-neutral) survival probability\(^3\) of the representative bank of the Libor panel, at time $t$ up to time $T$, is:

$$SP(t,T) = \frac{E^Q \left[ e^{-\int_t^T \lambda_s ds} \right]}{}$$

where $\tau^F$ is the time the default occurs, $\lambda_i$ is the (possibly stochastic) default intensity, which we assume independent from interest rates. We have also that $PD(t,T) = 1 - SP(t,T)$.

Assume for the moment we want to price a deposit starting in $t_s$ and expiring in $t_s + \tau$: the money is lent to a defaultable counterparty with a well defined survival probability $SP(t_s,t_s + \tau)$. In case of default, we suffer a percentage loss of the notional of the deposit market value equal to the Loss Given Default $Lgd$. To further simplify the notation, assume a unit notional.

The simply compounded risk-free (Ois) rate for the period $[t_s, t_s + \tau]$ is denoted by $R(t_s,t_s + \tau)$ and the simply compounded credit (basis) spread is denoted by $S(t_s,t_s + \tau)$. If $t_s > 0$ these are the simply compounded forward rates. The total Libor rate applied to the deposit is $L(t_s,t_s + \tau) = R(t_s,t_s + \tau) + S(t_s,t_s + \tau)$.

The credit spread represents a fair default risk premium over the risk free rate. As such, it is calculated in order to equate the discounted expected value of the risky deposit, under the risk neutral measure $Q$, (considering both cases of default during the contract lifetime and survival until maturity) to:

- the unit notional, if $t_s = 0$ (i.e.: if the deposit starts today);

\(^2\)Clearly, once the deal is struck and the counterparty is known, the exact credit risk borne by the lender can be different from the (average) credit risk of the representative bank.

\(^3\)It is likely superfluous stressing that we adopt a reduced form approach to default modelling.
the unit notional times the expected survival probability of the counterparty until \(t_s\), \(E^Q[1_{\tau^F > t_s + \tau}]\) if \(t_s > 0\) (i.e.: the deposit starts on a future date, and we weight the notional by the probability that the deposit actually starts, or that the counterparty survives at the start time).

Let us start by considering a spot starting deposit.

### 2.1 Spot Credit Spread

The equation to determine the spot credit spreads, assuming Recovery of Face Value (RFV), is:

\[
1 = E^Q \left[ D^D(0, \tau) \cdot \left[ 1 + (S(0, \tau) + R(0, \tau)) \cdot \tau \right] \right] = \\
= P^D(0, \tau) \cdot \left[ 1 + (S(0, \tau) + R(0, \tau)) \cdot \tau \right] \cdot SP(0, \tau) + (1 - Lgd) \cdot PD(0, \tau) = \\
\frac{1 + (S(0, \tau) + R(0, \tau)) \cdot \tau}{1 + R(0, \tau) \cdot \tau} \cdot SP(0, \tau) + \frac{(1 - Lgd) \cdot PD(0, \tau)}{1 + R(0, \tau) \cdot \tau} 
\]

(2)

We have indicated with \(D^D(t, T)\) the default risk-free discount factor from \(T\) to \(t\), and with \(P^D(t, T) = E^Q[D^D(t, T)] = \frac{1}{1 + R(t, T) \cdot \tau}\) the price in \(t\) of default risk-free zero-coupon bond expiring in \(T\).

By some simple algebra, we get from (2):

\[
S(0, \tau) = \frac{1}{\tau} \cdot \frac{(Lgd + R(0, \tau) \cdot \tau) \cdot PD(0, \tau)}{1 - PD(0, \tau)} 
\]

(3)

If we define the adjusted default probability as \(PD(0, \tau) = PD(0, \tau)/(1 - PD(0, \tau))\), then we can rewrite the spread as:

\[
S(0, \tau) = \frac{1}{\tau} \cdot (Lgd + R(0, \tau) \cdot \tau) \cdot PD(0, \tau) 
\]

(4)

### 2.2 Forward Credit Spread

In case of a forward start deposit, the forward credit spread, assuming again a Recovery of Face Value, is derived from the following equivalence:

\[
E^Q[1_{\tau^F > t_s}] = E^Q \left[ D^D(t_s + \tau, \tau) \cdot \left[ 1 + (S(t_s + \tau) + R(t_s + \tau)) \cdot \tau \right] \right] 
\]

(5)

Working out the expectations:

\[
SP(0, t_s) = \\
= P^D(t_s, t_s + \tau) \cdot \left[ 1 + (S(t_s, t_s + \tau) + R(t_s, t_s + \tau)) \cdot \tau \right] \cdot SP(0, t_s + \tau) + (1 - Lgd) \cdot (SP(0, t_s) - SP(0, t_s + \tau)) \\
= \frac{1 + (S(t_s, t_s + \tau) + R(t_s, t_s + \tau)) \cdot \tau}{1 + R(t_s, t_s + \tau) \cdot \tau} \cdot SP(0, t_s + \tau) + \frac{(1 - Lgd) \cdot (SP(0, t_s) - SP(0, t_s + \tau))}{1 + R(t_s, t_s + \tau) \cdot \tau} 
\]

(6)

In order to solve this equation, define the adjusted forward default probability as:

\[
PD(t_s, t_s + \tau) = \frac{E^Q[\tau^F > t_s]}{E^Q[\tau^F > t_s + \tau]} - 1 = \frac{SP(0, t_s)}{SP(0, t_s + \tau)} - 1 
\]

(7)

The forward credit spread \(S(t_s, t_s + \tau)\) is retrieved with some algebra:

\[
\frac{SP(0, t_s)}{SP(0, t_s + \tau)} \cdot (1 + R(t_s, t_s + \tau) \cdot \tau) = \\
1 + (S(t_s, t_s + \tau) + R(t_s, t_s + \tau)) \cdot \tau + (1 - Lgd) \cdot PD(t_s, t_s + \tau) 
\]

(8)

By expressing the LHS in terms of conditional forward default probabilities:

\[
PD(t_s, t_s + \tau) \cdot R(t_s, t_s + \tau) \cdot \tau + Lgd \cdot PD(t_s, t_s + \tau) = S(t_s, t_s + \tau) \cdot \tau \\
1 + R(t_s, t_s + \tau) \cdot \tau 
\]

This analysis is taken form Castagna and Fede [2].
Finally, we get:

\[ S(t_0, t_0 + \tau) = \frac{1}{\tau} \cdot (Lgd + R(t_0, t_0 + \tau) \cdot \tau) \cdot PD(t_0, t_0 + \tau) \]  

For typical values of interest rates, and relevant Libor tenors (i.e.: 1, 3, 6 and 12 months), we can safely assume that \( Lgd \approx Lgd + R(t_0, t_0 + \tau) \tau \). The spot and forward spreads can then be respectively written as:

\[ S(0, \tau) \approx \frac{Lgd}{\tau} PD(0, \tau) \]  

and

\[ S(t_0, t_0 + \tau) \approx \frac{Lgd}{\tau} PD(t_0, t_0 + \tau) \]  

Alternatively, if \( R(t_0, t_0 + \tau) \tau \) is not negligible, we can simply replace the original \( Lgd \) with \( Lgd^* = Lgd + R(t_0, t_0 + \tau) \tau \) and consider \( R(t_0, t_0 + \tau) \) a constant.

### 2.3 From Credit Spreads to Ois-Libor Basis

We now relax the assumption that the representative bank is exactly the same as a given bank, and we explicitly consider that it may change over time, mirroring the possible Libor panel’s changes.

We will introduce three different ways to model the modification of the Libor panel, that will produce three different models. We will dwell more on the first model, giving an intuitive representation of the panel transition process; the other two approaches are a variation that can be easily understood once one grasps the mechanics of the first one.

#### 2.3.1 Model 1: Stochastic Total Replacement of the Single Representative Bank

Assume we start with a given panel of banks characterised by a credit risk summarised in the spot Libor-Ois spread curve and referring to the representative bank at time \( t = 0 \). The credit spread curves are determined by the default probabilities commanded by an intensity process \( \lambda_H \), as in equation (1). At a future time \( \tau > 0 \) a change in the panel may occur: a new representative bank enters in the panel, replacing the bank currently entering it. This new bank has a credit quality determined by the default probabilities originated by another intensity process \( \lambda_L \), which starts exactly when the replacement occurs.

The representative bank may change over time due to its credit standing change (typically a worsening) or default. Any transition implies a reset of the process, meaning that the new process, referring to the new representative bank replacing the old one, will start exactly when the random transition event occurs.

Moving from a one-time to a continuous-time replacement process, we can generalise the idea outlined above in a rather straightforward fashion. Assume that we are at time \( t = 0 \), and that we are interested in determining the Libor forward spread for a deposit starting in \( t_0 \) and expiring in \( t_0 + \tau \), as indicated in Figure 5. At time \( t_0 \) the panel will be the one at time 0 with probability \( w_{0,0}(t_0) \); during the period \([0, t_0]\) a continuous replacement process takes place: at each time \( \xi_i \), for \( i > 0 \), a new panel can replace the original one, and the probability that this is the panel existing at the start of the deposit in \( t_0 \) is indicated by \( w_{0,0}(t_0, \xi_i) \). To each new panel, corresponding to a given representative bank, is associated a specific default risk, commanded by a default intensity process starting in \( \xi_i \).

Hence, at time \( t_0 \), loosely speaking, the representative bank’s default probability will be a weighted average of all the default probabilities of the representative banks that can form the panel by the time \( \xi_i \) on.

It is important to highlight the fact that the deposit counterparty, i.e.: the borrower bank, is a specific member of the interbank population that is implicitly assumed to be an infinity of banks that can replace the defaulted, or credit deteriorated, banks included in the current and future panels. Whenever there is a credit standing transition or a default, the current representative agent changes and the whole population changes accordingly. This transition in overall population characteristics is equivalent to a replacement of the representative agent with a new kind of representative agent.

It is assumed that all banks have mutually independent default intensity dynamics. Therefore, given this assumption of independence and infinite population, we imply that there will always
be a bank (embodied in the representative bank) to which the Libor rate can be applied when it asks to borrow money. In other words, although the Libor panel is made of defaultable entities, the replacement process (jointly with the above mentioned assumptions) ensures that the process of the Libor-Ois basis never stops, and that there is always the opportunity to lend money to a Libor bank.

When we monitor the credit risk in a forward starting deposit with a specific representative bank, we always consider the possibility that it might go bankrupt before the contract inception in $t_s$ and this will be accounted for in the specific representative bank’s credit spread by using the relevant forward default probabilities. In case a specific representative bank defaults (i.e.: the panel stops existing and a replacement occurs), we will move our monitoring to another representative bank which is independent from the previously monitored one. Given our assumptions we can rest assured that we will always find a new representative bank to monitor.

We will provide in next section the formulae for Forward Libor-Ois basis curves referring to general version of this model: it allows for a change in the Libor panel after a total replacement of the existing representative bank with a new representative bank that can be either of type $L$ or $H$.

Forward Libor-Ois basis curves

Whenever a series of replacement events occurs between $t = 0$ and $t = t_s$, only the last of these events is relevant, since on every replacement the previous default intensity process stops and is replaced by the default intensity process of the new representative bank replacing the old one. Once the last replacement occurs, we can use the credit spread equations defined in the beginning of this section, relating the basis to the forward default probability under the last extracted $\lambda$ process.

Assume the last replacement time is $\xi$, so that the new default intensity process $\lambda$ starts exactly in $\xi$. Consider two cases: in the first the last new bank will be of type $H$, in the other it will be of type $L$. Given the assumption that the functional form of the intensity process $\lambda$ between switching events does not change, we have that the forward default probability will be simply shifted in time by $\xi$: $\text{PD}_z(t_s - \xi, t_s - \xi + \tau)$ (with $z \in \{H, L\}$), where

$$\text{PD}_z(t_s - \xi, t_s - \xi + \tau) = \frac{\text{SP}_z(0, t_s - \xi)}{\text{SP}_z(0, t_s - \xi + \tau)} - 1 \quad (12)$$
We also trivially have \( SP_2(t_z - \xi, t_s - \xi + \tau) = 1 - PD(t_z - \xi, t_s - \xi + \tau) \).

Additionally, denote the conditional credit spread, under the replacement conditions, as \( S_H(t_s - \xi, t_s - \xi + \tau) \) and \( S_L(t_s - \xi, t_s - \xi + \tau) \) respectively. Making use of the approximation introduced above, the credit spread is:

\[
S_L(t_s - \xi, t_s - \xi + \tau) = \frac{L_{gd}}{\tau} \cdot PD(t_z - \xi, t_s - \xi + \tau) \tag{13}
\]

where \( PD(t_z - \xi, t_s - \xi + \tau) = SP_2(0, t_s - \xi) / SP_2(0, t_s - \xi + \tau) - 1 \).

So far we showed the calculations for a credit spread conditioned on a specific last replacement event \( \xi \in (0, t_s) \). We need to integrate for all the possible \( \xi \)'s in the interval \((0, t_s]\), bearing in mind that two last replacement time \( \xi \)'s are obviously mutually exclusive. Therefore, to compute the unconditional Libor-Ois basis, we have to consider these three general possibilities:

1. the representative bank is never replaced, therefore we will calculate the basis as the one of the initial bank;
2. the representative bank is replaced at least once and in the last replacement the new bank is of type \( H \);
3. the representative bank is replaced at least once and in the last replaced the new bank is of type \( L \).

The first case has probability: \( w_{H,0}(0, t_s) \), and the spread is:

\[
S_1(t_s, t_s + \tau) = w_{H,0}(t_s) \cdot S_H(t_s, t_s + \tau) \tag{14}
\]

The weight \( w_{H,0}(0, t_s) \), as well as the other weights in the following formulae, are derived in Appendix A\(^5\).

In the second case, the spread is calculated by integrating over all admissible \( \xi \)'s the conditional spread \( S_H(t_s - \xi, t_s - \xi + \tau) \) multiplied by the probability density function \( w_{HH}(t_s, \xi) \).

\[
S_2(t_s, t_s + \tau) = \int_0^{t_s} w_{HH}(t_s, \xi) \cdot S_H(t_s - \xi, t_s - \xi + \tau) d\xi \tag{15}
\]

In the third case the spread is calculated similarly to the second case:

\[
S_3(t_s, t_s + \tau) = \int_0^{t_s} w_{HL}(t_s, \xi) \cdot S_L(t_s - \xi, t_s - \xi + \tau) d\xi \tag{16}
\]

Recall that the probabilities \( w_{z_1,z_2}(t, T) \) refer to the last replacement from \( z_1 \) to \( z_2 \) occurring between \( t \) and \( T \): they implicitly contain also all the possible replacements from the two types of representative bank occurring before the last one.

The unconditional forward Libor-Ois basis is simply the sum of the three terms above, since they are mutually exclusive and they are already weighted for their respective probabilities:

\[
S_{Libor}(t_s, t_s + \tau) = S_1(t_s, t_s + \tau) + S_2(t_s, t_s + \tau) + S_3(t_s, t_s + \tau) \tag{17}
\]

Marginal forward Libor-Ois basis p.d.f.

To derive the forward Libor-Ois basis marginal p.d.f., we need to condition it on:

- a certain state \( z \in \{H, L\} \)
- last replacement event in \( \xi \in (0, t_s) \)
- survival until \( t_s - \xi \)

We need to derive the complete density, accounting for all possible \( \xi \in (0, t_s) \). To this end, consider the three cases:

1. No replacement of the representative agent occurs in \((0, t_s)\)

2. One or more replacements occur in \((0, t_s)\), with a last replacement event in \(\xi \in (0, t_s)\) collapsing in state \(H\)

3. One or more replacements occur in \((0, t_s)\), with a last replacement event in \(\xi \in (0, t_s)\) collapsing in state \(L\)

The respective probabilities are:

1. \(w_{H,0}(t_s)\)
2. \(w_{HH}(t_s, \xi)\)
3. \(w_{HL}(t_s, \xi)\)

The p.d.f. is:

\[ g^Q(S_{Libor}, 0, t) = g^Q_1(S_{Libor}, 0, t) + g^Q_2(S_{Libor}, 0, t) \]

where the functions \(g^Q_1(.)\), \(g^Q_2(.)\) and \(g^Q_3(.)\) are given in Appendix A.

### 2.3.2 Model 2: Stochastic Partial Replacement with Detailed Libor Panel

In the Model 2 we extend the idea of Model 1 by allowing for a detailed description of the initial Libor panel. In more detail, assume that the panel is composed by \(N\) banks\(^6\). They can be both of type \(H\) and \(L\), typically with a mix at the observation date containing more of the former if Libor-Ois curves are humped.

Each bank in the panel can be replaced by new banks of both types in the future; the credit spread of these banks is commanded by an intensity process that starts at the time the replacing banks enter in the panel. The replacement process is modelled in the same way as Model 1, by a continuous time Markov chain.

The main difference between Model 1 and Model 2 is that in the latter we consider the actual number of banks entering in the Libor panel, although they can only be of two types. The modifications of the panel can occur for any of the \(N\) banks at random future times, contrarily to Model 1, in which the (one) representative bank can be replaced at future times by another representative bank, thus completely renewing the composition of the panel.

**Forward Libor-Ois basis curves**

If we denote the \(i\) member’s initial state as \(z_i(0) \in \{H, L\}\), we have two kinds of random variables depending on \(z_i(0)\).

Each random variable \(\tilde{S}_i\) will follow dynamics according to Model 1, with initial state \(z_i(0)\). We divide the panel members in two subsets \(Z_H = \{i \mid z_i(0) = H\}\) and \(Z_L = \{i \mid z_i(0) = L\}\). Then the Libor-Ois basis random variable may be expressed as:

\[ \tilde{S}_{Libor}(t_s, t_s + \tau) = \frac{1}{N} \left( \sum_{i \in Z_L} \tilde{S}_i(t_s, t_s + \tau) + \sum_{j \in Z_H} \tilde{S}_j(t_s, t_s + \tau) \right) \quad (19) \]

Denote with \(S_{M_{1,H}}(t_s, t_s + \tau)\) and \(S_{M_{1,L}}(t_s, t_s + \tau)\) the value of a forward Libor-Ois basis calculated according to Model 1 with initial state \(H\) and \(L\) respectively. Suppose that there are \(m\) members in \(Z_H\) and \(N - m\) members in \(Z_L\). Thus the expectation of \(\tilde{S}_{Libor}\) will be the weighed average of the forward Libor-Ois basis above defined:

\[ S_{Libor}(t_s, t_s + \tau) = \frac{N - m}{N} \cdot S_{M_{1,L}}(t_s, t_s + \tau) + \frac{m}{N} \cdot S_{M_{1,H}}(t_s, t_s + \tau) \quad (20) \]

\(^6\)At the time of writing, the USD Libor panel is made of 18 banks; the EUR Euribor panel is made of 25 banks.
Marginal forward Libor-Ois basis p.d.f.

To find out the density of $\tilde{S}_{\text{Libor}}(t_s, t_s \tau)$, we need to calculate the p.d.f. of the following random variable:

$$\tilde{S}_{\text{Libor}}(t_s, t_s \tau) = \frac{1}{N} \left( \sum_{i \in Z_L} \tilde{S}_i(t_s, t_s \tau) + \sum_{j \in Z_H} \tilde{S}_j(t_s, t_s \tau) \right)$$

(21)

Suppose that the spread of the bank $i$ is independent from the spread of the bank $j$ for every $i \neq j$. The density we are looking at is simply the convolution of the densities $\tilde{S}_{i \in Z_L}(t_s, t_s \tau)$ and $\tilde{S}_{j \in Z_H}(t_s, t_s \tau)$. The forward Libor-Ois basis $S$ is given by the convolution of two components: i) the p.d.f. of the weighted sum of the $S_{0 \in Z_L}(t_s, t_s \tau)$ and $S_{1 \in Z_H}(t_s, t_s \tau)$ for every $S_{i \in Z_L}(t_s, t_s \tau)$ and $S_{j \in Z_H}(t_s, t_s \tau)$.

Denote with $g^Q_{M_L}(S, t_s, t_s \tau)$ and $g^Q_{H}(S, t_s, t_s \tau)$ the Libor-Ois basis marginal density according to Model 1, with initial states $H$ and $L$ respectively. These are the respective p.d.f. for $Z_H(t_s, t_s \tau)$ and $Z_H(t_s, t_s \tau)$.

If there are $m$ members in $Z_H$ and $N-m$ members in $Z_L$, the marginal density of the Libor-Ois basis is given by the convolution of two components: i) the p.d.f. of the weighted sum of the $m$ members in group $Z_H$ and ii) the p.d.f. of the weighted sum of $N-m$ members in group $Z_L$. Each member is equally weighted by $1/N$.

$$S_{\text{Libor}}(y, t_s, t_s \tau) =$$

$$= \left[ \left( N \cdot g^Q_{M_L}(N \cdot \tilde{S}, t_s, t_s \tau) \right) \right] \cdot \cdots \left( N \cdot g^Q_{H}(N \cdot \tilde{S}, t_s, t_s \tau) \right)$$

$$\ast \left[ \left( N \cdot g^Q_{M_L}(N \cdot \tilde{S}, t_s, t_s \tau) \right) \right] \cdot \cdots \left( N \cdot g^Q_{H}(N \cdot \tilde{S}, t_s, t_s \tau) \right)$$

(22)

where $\ast$ denotes the convolution operator.

2.3.3 Model 3: Continuous Time Deterministic Replacement Process of the Single Representative Bank

The Model 3 for the Libor-Ois spread hinges on the assumption that new replacing representative banks gradually replace the representative banks entering the initial Libor panel. So we have two main differences between Model 1 and Model 3: i) the initial panel can be a combination of $H$ and $L$ type banks in Model 3, whereas it was a panel made by a single type representative bank in Model 1; ii) in Model 3 the replacement is not total, as in Model 1, but only a fraction of old representative banks can be replaced by new $H$ and $L$ type banks; finally iii) the replacement process occurs continuously and in a deterministic fashion in Model 3, contrarily to the Markov chain process in Model 1. The technical details of the transition mechanics are in the Appendix B.

Forward Libor-Ois basis curves

The forward Libor-Ois basis $S_{\text{Libor}}(t_s, t_s \tau)$ in this model is the sum of two components:

1. the contribution of the initial panel, denoted by $S_1(t_s, t_s \tau)$;
2. the integral of the contributions of all the new banks that enter the panel in $(0, t_s)$, denoted with $S_2(t_s, t_s \tau)$.

Denote with $S(t_s, t_s \tau) = S_H(t_s, t_s \tau)S_L(t_s, t_s \tau)$ the vector containing the $H$ and $L$ single static counterparty forward credit spreads. The first component is given by:

$$S_1(t_s, t_s \tau) = (B(0, t_s) \cdot w_0)^T \cdot S(t_s, t_s \tau)$$

(23)

where $B(0, t_s)$ is the decay matrix, defined in (85), Appendix B, and $w_0$ is the initial Libor panel composition vector.
The second component is given by:

\[ S_2(t_s, t_s + \tau) = \int_0^{t_s} (\varphi(u, t_s))' \cdot S(t_s - u, t_s - u + \tau) \cdot du \]  

(24)

where \( \varphi(u, t_s) \) is the new bank weight density vector (with \( H \) and \( L \) components) defined in equation (83), Appendix B.

Finally, the complete forward Libor-Ois basis is given by:

\[ S_{Libor}(t_s, t_s + \tau) = S_1(t_s, t_s + \tau) + S_2(t_s, t_s + \tau) \]  

(25)

The formulae for the single components are in the Appendix.

### Marginal forward Libor-Ois basis p.d.f.

Let \( S_H(t_s, t_s + \tau) \sim p^O_H(S_H, t_s, t_s + \tau) \) and \( S_L(t_s, t_s + \tau) \sim p^O_L(S_L, t_s, t_s + \tau) \), where \( p^O_2 \) is the risk neutral marginal p.d.f of a credit spread for a z-type bank, for a deposit starting in \( t_s \) and maturing in \( t_s + \tau \). Then:

\[ S_1(t_s, t_s + \tau) \sim g^O_1(S_1, t_s, t_s + \tau) g^O_1(S_1, t_s, t_s + \tau) = \]

\[ = \left[ \frac{1}{b_H(0, t_s)} w_H(0) \right] \cdot p^O_H \left( \frac{S_1(t_s, t_s + \tau)}{b_H(0, t_s)} w_H(0), t_s, t_s + \tau \right) \]

(26)

\[ \ast \left[ \frac{1}{b_L(0, t_s)} w_L(0) \right] \cdot p^O_L \left( \frac{S_1(t_s, t_s + \tau)}{b_L(0, t_s)} w_L(0), t_s, t_s + \tau \right) \]

where * denotes the convolution operator.

Then the complete spread has the following p.d.f.: 

\[ S_{Libor}(t_s, t_s + \tau) \sim g^O_1(S_{Libor} - S_2(t_s, t_s + \tau), t_s, t_s + \tau) \]

(27)

As explained in Appendix B, \( S_2(t_s, t_s + \tau) \) has infinitesimal variance and therefore is a deterministic process. The detailed formula is given in Appendix B as well.

### 3. Libor Caplet&Floorlet Valuation with Stochastic Basis

The framework outlined above allows to retrieve the marginal densities for the Libor-Ois basis in each of the three models analysed. It is then possible, under the assumption of the Libor-Ois basis independent from the corresponding Ois rate, the sum of both being the Libor rate.\(^7\)

Let us start by considering a deterministic and constant additive spread \( \hat{S} \): then a Libor caplet would be equivalent to a caplet on an Ois rate with adjusted strike \( \hat{K} = K - \hat{S} \). Let \( R(t_s, t_s + \tau) \) be the forward Ois rate between \( t_s \) and \( t_s + \tau \), observed at time \( t \). A caplet on Ois rate \( L(t_s, t_s + \tau) = R(t_s, t_s + \tau) + S \) has a pay-off at the natural expiry in \( t_s \) equal to:

\[ \text{Caplet}_{Libor} (L(t_s, t_s + \tau), K, t_s, t_s + \tau) = \]

\[ = \max [R(t_s, t_s + \tau) + S - K, 0] = \]

\[ = \max [R(t_s, t_s + \tau) - (K - S(t_s, t_s + \tau))], 0] = \]

\[ = \text{Caplet}_{Ois} (R(t_s, t_s + \tau), K - \hat{S}, t_s, t_s + \tau) \]

At time \( t \), the caplet is worth \( \text{Caplet}_{Ois} (R(t_s, t_s + \tau), K - \hat{S}, t_s, t_s + \tau) \) and can be computed by any available model commonly adopted in practice, e.g.: a Black formula.

If we consider a stochastic credit spread and assume its independence from Ois rates, we can simply evaluate the Libor caplet as the Ois caplet above conditioned on all admissible values of \( \hat{S} \). We may then express the value of a Libor caplet as the convolution between the Ois caplet (as a

\(^7\)Similar general formulae are given also in Mercurio [5].
TABLE 1: Ois rates. The underlying short rate follows a CIR process with $r_0 = 0.80\%$, $\kappa_{\text{Ois}} = 1.5$, $\theta_{\text{Ois}} = 1.00\%$ and $\sigma_{\text{Ois}} = 5.00\%$.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1M</th>
<th>3M</th>
<th>6M</th>
<th>1Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.81%</td>
<td>0.83%</td>
<td>0.86%</td>
<td>0.90%</td>
</tr>
<tr>
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<td>0.96%</td>
<td>0.97%</td>
<td>0.98%</td>
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<tr>
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<tr>
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<td>1.00%</td>
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<td>1.00%</td>
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</tr>
</tbody>
</table>

TABLE 2: Parameters for models with constant replacement function of the strike $K$ and the marginal basis density $g^Q(S, t_s, t_s + \tau)$, whose explicit formula is given for each of the three models:

$$
\text{Caplet}_{\text{Libor}}(L(t_s, t_s + \tau), K, t_s, t_s + \tau) = \text{Caplet}_{\text{Ois}}(R(t_s, t_s + \tau), K, t_s, t_s + \tau) \ast g^Q(K, t_s, t_s + \tau) = \int_{-\infty}^{+\infty} \text{Caplet}_{\text{Ois}}(R(t_s, t_s + \tau), K - S, t_s, t_s + \tau) \cdot g^Q(S, t_s, t_s + \tau) \cdot dS
$$

where $\ast$ indicates the convolution operator, setting as the convolution domain $K \in \mathbb{R}$.

4. A Specification of the Model with CIR Intensity Dynamics

As we have mentioned above, the framework we have sketched is modular, in the sense that, under the stated assumptions, we can choose any dynamics for the intensity processes for $H$- and $L$-type banks, and thus specify the Models 1, 2 or 3 we have analysed above for the Libor-Ois basis.

Besides, we can choose any dynamics for the (risk-free) Ois rate and then come up with a full specification for the Libor rate dynamics that will allow for the valuation of Libor derivatives, including caps & floors and swaptions.\(^8\)

In what follows we will specify the default intensity dynamics as CIR processes\(^9\) and we will show the basis curve it is possible to obtain by the three models of Libor-Ois basis.

4.1 Forward Credit Spreads

Assume a bank of type $z$ can go defaulted according to a jump process commanded by an intensity whose dynamics - under the risk neutral measure $Q$ - follows a CIR process of the type:

$$
d\lambda_z(t) = \kappa_z (\theta_z - \lambda_z(t)) \cdot dt + \sigma_z \sqrt{\lambda_z(t)} \cdot dW^Q_{z,t} \quad (29)
$$

\(^8\)We have not studied the evaluation of swaptions in this work, but it is possible in the outlined framework.

\(^9\)See Cox, Ingersoll and Ross [3].
where \( z \in \{ H, L \} \) is a label variable indicating whether the counterparty is of the H or L type (high or low credit risk respectively). The initial condition is \( \lambda_0 = \lambda_z(0) \).

Since the CIR process belongs to the affine exponential family, forward credit spreads may be explicitly derived. Given the survival probability:

\[
\text{SP}_z(t, T) = \mathbb{E} \left[ \exp \left( - \int_0^t \lambda_z(u) du \right) \right] = A_z(t, T) \exp \left( - B_z(t, T) \lambda_z(0) \right) \quad (30)
\]

\[
\gamma_z = \sqrt{\kappa_z^2 + 2\sigma_z^2 v_z} = \frac{4\kappa_z \theta_z}{\kappa_z^2} A_z(T, S) = \left[ \frac{2\gamma_z \exp \left( (\kappa_z + \gamma_z) (S - T)/2 \right)}{2\gamma_z + (\kappa_z + \gamma_z) (\exp \left( [S - T]/\gamma_z \right) - 1)} \right]^{\gamma_z/2} B_z(T, S) = \left( \frac{2 \exp \left[ \gamma_z (S - T) \right] - 1}{2\gamma_z + (\kappa_z + \gamma_z) (\exp \left[ \gamma_z (S - T) \right] - 1)} \right)^{\gamma_z/2} \quad (31)
\]

the forward credit spreads are:

\[
S_z(t_s, t_s + \tau) = \frac{\text{Lgd}}{\tau} \cdot \text{PD}(t_s, t_s + \tau) = \frac{\text{Lgd}}{\tau} \cdot \left[ \frac{A_z(0, t_s) \exp \left( -B_z(0, t_s) \lambda_z(0) \right)}{A_z(0, t_s + \tau) \exp \left( -B_z(0, t_s + \tau) \lambda_z(0) \right)} \right]^{\gamma_z/2 - 1} \quad (32)
\]

**Practical Examples**

Having specified the dynamics of the intensity \( \lambda(t) \) as a CIR process, we are now able show the time structures for the Libor-Ois basis generated within our framework. The aim of all the models is to accurately reproduce the humped shape of the real data (see Figure 1): it means that our model is designed to be flexible enough to reproduce the hump in a realistic time interval, to match the slope of the time structure and to replicate the spot (deposits’) Libor-Ois spread.

Assume we set the parameters of the CIR intensity process for the \( H \) and \( L \) type banks, in each of the 3 models, as shown in the table 2. We can then check which type of shapes for the Libor-Ois basis term structure the 3 models generate. Moreover, we test also the flexibility of the model by introducing time dependent parameters in the transition processes and in the exit intensity from the panel.

The starting Ois rates’ term structure is also needed: we generate a curve by a CIR model for the short rate whose parameters are chosen such that they fit best the market quotes dealing on November 1st, 2014. In table 1 we show the term structures of the forward Ois for the 1M, 3M, 6M and 1Y tenor.

The results are shown in figures 6(a,c,e), where we used fixed parameters for the exit intensity from the panel and for the transition dynamics specific to each model. Note that in the Model 2 (Stochastic Partial Replacement with Detailed Libor Panel) we considered a panel of 25 banks, such that 15 started as \( H \) type. Moreover, in the Model 3 (Continuous Time Deterministic Replacement Process of the Single Representative Bank) we chose a starting panel entirely composed of \( H \) type banks, that is \( w_H(0) = 1 \) and \( w_L(0) = 0 \).

In figure 6(b,d,f) we show the Libor-Ois basis term structure when allowing for time dependent parameters of the panel exit intensity and transition dynamics. The most accurate model seems to be the second one, that is the model with stochastic partial replacement with detailed Libor Panel. As a matter of fact, such a time structure does not suffer the initial spike and moreover the hump correctly cover the interval between spot date and 10-ish years, switching from a concave to convex slope.

### 4.2 Credit Spread Marginal p.d.f.

Given a CIR specification for the dynamics of \( \lambda_z(t) \), we wish to calculate the marginal p.d.f. of the credit spread \( \hat{S}_z(t_s, t_s + \tau) \), which is indicated as \( p^{\text{U}}_z(\hat{S}_z(t_s, t_s + \tau)) \).
FIGURE 6: An example of forward basis curve with constant and time variant parameters.
Since \( \tilde{S}_z(t_s, t_s + \tau) = \frac{1}{\tau} \left( \frac{\text{SP}(0, t_s)}{\text{SP}(0, t_s + \tau)} - 1 \right) \), if we define the random variable \( x = \frac{\text{SP}(0, t_s + \tau)}{\text{SP}(0, t_s)} \), we may equivalently say that the relationship between \( \tilde{S}_z(t_s, t_s + \tau) \) and \( x \) is:

\[
\tilde{S}_z(t_s, t_s + \tau) = \frac{1}{\tau} \left( \frac{1}{x} - 1 \right)
\]  

(33)

Note that the previous relation is a deterministic, invertible and differentiable function. So, if we calculate the risk-neutral marginal p.d.f. of \( x \) first, which we will denote as \( f^Q_z(x, t_s, t_s + \tau) \), we are able to deduce from it \( p^Q_z(S_z, t_s, t_s + \tau) \). The details are explained the Appendix A.

The CIR process belongs to the Affine Exponential Family, therefore the random variable:

\[
\tilde{SP}_z(t_s, t_s + \tau) = E^Q \left[ \exp \left( - \int_{t_s}^{t_s + \tau} \lambda_z(u) du \right) \right] = A(t_s, t_s + \tau) \exp \left( - B(t_s, t_s + \tau) \lambda_z(t_s) \right)
\]

may be expressed in terms of the \( \lambda_z(t_s) \) random variable.

Since we wish to derive the p.d.f. of \( x \), which is a ratio of survival probabilities, let us define \( G^t_z \) the probability measure associated with the numeraire \( \tilde{SP}(0, t_s) \). The associated Radon-Nikodym derivative is:

\[
\frac{dQ}{dG^t_z}(t) = \frac{1}{\text{SP}(t, t_s)}
\]

(34)

Using conditional expectations:

\[
f^Q_z(x) \cdot dx = E^Q \left[ 1_{\tilde{SP}_z(t_s, t_s + \tau) \in (x, x + dx)} \right] F_{t_s} = E^{G^t_z} \left[ 1_{\tilde{SP}_z(t_s, t_s + \tau) \in (x, x + dx)} \right] \frac{dQ}{dG^t_z} \bigg| F_{t_s}
\]

(35)

therefore to obtain our result we may equivalently switch from \( Q \) to the \( G^t_z \) measure.

Let \( v \leq t \leq t_s \) and define these auxiliary functions:

\[
\begin{align*}
\varphi_z &= \frac{k_z + \gamma_z}{\sigma_z^2} \\
q_z(t, v) &= 2 \left[ \rho_z(t - v) + \varphi_z + B_z(t, t_s) \right] \\
\delta_z(t, v, \lambda_z(v)) &= 4 \left( \rho_z(t - v) \right)^2 \cdot \lambda_z(s) \cdot e^{-\gamma_z(t-v)} \cdot \left( \frac{q(t, s)}{q(t, s)} \right)
\end{align*}
\]

Under the forward measure \( G^t_z \), the distribution of \( \lambda_z(t) \) conditional on \( \lambda_z(v) \) is given by:

\[
p^G_{\lambda_z(t) \mid \lambda_z(v)}(x) = q_z(t, v) \cdot p_{x^2(v, z, \lambda_z(t), \lambda_z(s))} \cdot (q_z(t, v) \cdot x)
\]

(36)

where \( p_{x^2(v, z, \lambda_z(t), \lambda_z(s))} \) denotes the marginal p.d.f. of a Non-Central Chi-Squared random variable with \( v \) degrees of freedom and non-centrality parameter \( \delta \).

Given the \( G^t_z \) p.d.f. of \( \lambda_z(t) \), by inverting this relation we are able to derive the p.d.f. of \( A_z(t_s, t_s + \tau) \exp \left( - B_z(t_s, t_s + \tau) \lambda_z(t_s) \right) \) under \( G^t_z \). Set \( x = \tilde{SP}_z(t_s, t_s + \tau) \).

\[
\lambda_z(t_s) = \frac{1}{B_z(t_s, t_s + \tau)} \cdot \log \left( \frac{A_z(t_s, t_s + \tau)}{x} \right)
\]

(37)

This is an invertible and differentiable function of \( x \). Its first derivative is:

\[
\frac{d\lambda_z(t_s)}{dx} = \frac{1}{B_z(t_s, t_s + \tau) \cdot x}
\]

(38)

Using the classical probability result for the p.d.f. of an invertible and differentiable function of a random variable:

\[
f^Q_z(x, t_s, t_s + \tau) = \left| \frac{d\lambda_z(t_s)}{dx} \right| \cdot q_z(t_s, v) \cdot p_{x^2(v, z, \lambda_z(t_s), \lambda_z(s))} \left( q_z(t_s, v) \cdot \lambda_z(t_s) \right)
\]

(39)
We finally get:
\[
\begin{align*}
 f^{Q}_{z}(x, t_s, t_s + \tau) &= \\
&= \frac{1}{B_z(t_s, t_s + \tau)} \cdot x \cdot q_z(t_s, \nu) \cdot p_{\chi^2(\nu, \phi_z(t_s, \nu))} \left( q_z(t_s, \nu) \cdot \frac{1}{B_z(t_s, t_s + \tau)} \cdot \log \left( \frac{A_z(t_s, t_s + \tau)}{x} \right) \right)
\end{align*}
\]

Once we have \( f^{Q}_{z}(\cdot) \), we are able to calculate the p.d.f. of the credit spread \( \tilde{p}^{Q}_{z} \) according to (33)
\[
\tilde{p}^{Q}_{z}(\tilde{S}, t_s, t_s + \tau) = \frac{\tau}{\text{Lgd}} \cdot \frac{f^{Q}_{z}(\Theta(\tilde{S}), t_s, t_s + \tau)}{\left( \frac{\tau}{\text{Lgd}} \tilde{S} + 1 \right)^2}
\]  

\( (40) \)

This formula is explained in details in the Appendix. We are now able to make it specific to any of the 3 models:

1. for the first model the p.d.f. is given by the formula (76) in Appendix A;
2. once we have the p.d.f. for the first model, we easily deduce the p.d.f. for the second model applying (22);
3. the p.d.f. for the third model is explicitly shown in formulas (103) and (104), Appendix B.
FIGURE 8: An example of volatility surfaces for a caplet with expiry 10 years and tenor 6M. Ois rate is modelled by Black model with implied volatility equal to 30%.
FIGURE 9: P.d.f. for Libor-Ois basis referring to 6M tenor
Practical Examples

Given the densities for each model, the Libor Caplet will be a consequence of equation (28). We use the same data for the Ois rates and the Libor-Ois spreads as above. For the Ois forward rates we also assume that they are lognormally distributed with one constant volatility set at 30%. Please note that we should have used a CIR model for the Ois rates also to evaluate caplets, to be consistent with the way we generated the Ois forward curves. Nonetheless, the purpose of this section is to show which is the impact on the smile shape introduced by the Libor-Ois basis models we have introduced. For the same reason, we compare also the volatility smiles produced by the Libor-Ois models with the smile generated by a simply displaced Lognormal model, with displacement set equal to the relevant forward Libor-Ois basis, assumed to be constant.

We are then able to calculate the implied volatility for each of the models as shown in Figures 7 and 8 for the 6M tenor. Implied volatility smiles and implied volatility surface for other tenors (1M, 3M, 1Y) are shown in Appendix C.

We are also able to plot the p.d.f. for each model and for each tenor. Figure 9 shows the p.d.f. for the 6M tenor at different maturities for all the models. The other tenors present almost identical shapes, by differing from the 6M only for the center of the peaks, since each tenor’s p.d.f. is centered on its own forward rate. With the chosen set of parameters, the p.d.f. of the third model implies a practically nil volatility. The other distribution for the other two models, both in the constant and time variant parameters, are multi-modal, due to the replacement mechanism.

5. Conclusions

In the present paper we provided a framework that is based entirely on micro-founded inference and credit risk arguments: all of the assumptions, dynamics and parameters derive from considerations on how market practitioners typically deal with the Libor-Ois basis.

The framework is flexible enough to capture the features of the Libor-Ois basis quoted on the market. In fact it would be able to reproduce a broad variety of complex basis term structures given the right time structure for the replacement parameters. We consider such a feature one of the strengths of our setup. Since the replacement parameters in our models have a straightforward interpretation, they provide also a simple tool to analyze and interpret Libor-Ois basis expectations implied in market quotes.

Furthermore, the present framework addresses the issue of market illiquidity for some regions of the term structure. In fact, in most cases available market quotes are not sufficient to cover the whole Libor-Ois basis term structure for a given tenor: practitioners can use our framework to deduce the illiquid parts of the curves for all tenors from the quotes of actively traded instruments. In our practical examples, the liquid 3M and 6M indexed instruments were sufficient to reproduce the entire set of curves.

We noticed a rather high probability of replacement of the Libor panel: this is likely due to the fact that not actively traded assets entered in the calibration process. This factor leads to high replacement intensity, which in turn leads to small volatility for Libor-Ois basis distributions.

If we consider only actively traded instruments, we suspect that reasonable replacement parameters will be sufficient for calibration. Our framework could then be used to provide a more suitable interpolation and extrapolation method for the entire Libor-Ois basis term structure.
References


