Pricing of Collateralized Derivatives Contracts when More than One Currency are Involved: Liquidity and Funding Value Adjustments *

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1 Introduction

In this paper we try and complete the analysis conducted in Castagna [1] by investigating the valuation of collateralized derivative contracts when more than one currency are involved. This can happen for three reasons:

1. The contract's pay-off is denominated in some currency **YYY** but collateral is posted in another currency **XXX**;
2. The contract is written on an FX rate;
3. The contract's pay-off depends on assets or market variables denominated in different currencies (*e.g.*: a cross currency interest rate swap).

In theory we could have many currencies involved, but in what follows we restrict our analysis to the case when only two currencies have to be considered. We will analyse all the cases enumerated above and we will define the liquidity value adjustments and funding value adjustments for collateralized contracts.

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2 Contracts Collateralized in a Currency Other than the Pay-off's Currency

Assume we have to value a contract whose underlying asset follows a dynamics of the type:

$$dS_t = (\mu_t - y_t)S_t dt + \sigma_t S_t dZ_t$$

The underlying has a continuous yield of y_t and a volatility σ_t, and it is denominated in a currency, that we name "domestic", **YYY**. There exists also a foreign currency **XXX** and the exchange rate $\mathcal{X} =$ **XXXYYY**[1] follows the dynamics:

$$d\mathcal{X}_t = \eta_t \mathcal{X}_t dt + \nu_t \mathcal{X}_t dW_t$$

with $dW_t dZ_t = \rho dt$.

We want to replicate a derivative contract V written on S, which is collateralized continuously in the **XXX** currency instead of **YYY**, as it would be normally the case. Following the same approach as in Castagna [1] (whom we refer to for details), we build a portfolio replicating bot the underlying and the collateral account.

The contingent claim dynamics is derived via the Ito's lemma:

$$dV_t = \mathcal{L}^{\mu} V_t + \sigma_t S_t \frac{\partial V_t}{\partial \mathcal{X}_t} dZ_t$$

where we used the operator $\mathcal{L}^a \cdot$ defined as:

$$\mathcal{L}^a \cdot = \frac{\partial \cdot}{\partial t} + a_t S_t \frac{\partial \cdot}{\partial S_t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 \cdot}{\partial S_t^2}$$

Besides, we will set also $\Delta_t = \frac{\partial V_t}{\partial S_t}$ in what follows. The cash collateral account dynamics is defined as

$$dC\mathrm{f}_t = \gamma d\left(\frac{V_t}{\mathcal{X}_t}\right) + c\mathrm{f}_t C\mathrm{f}_t dt$$

The account is denominated in **XXX**, if earns the collateral rate $c\mathrm{f}_t$ collateral rate; $r\mathrm{f}_t$ is the funding/investment rate in **YYY**. For the collateral account is also true that:

$$\begin{aligned}
C\mathrm{f}_0 &= \gamma(V_0/\mathcal{X}_0) \\
C\mathrm{f}_{T^-} &= E^Q\left[\int_0^{T^-} e^{-\int_u^{T^-} c\mathrm{f}_v dv}\gamma d(V_u/\mathcal{X}_u)\right] \\
C\mathrm{f}_T &= 0
\end{aligned}$$

The **YYY** bank cash account evolution is deterministic and equal to:

$$dB_t = r_t B_t dt \tag{1}$$

and the **XXX** bank cash account evolution is:

$$dB\mathrm{f}_t = r\mathrm{f}_t B\mathrm{f}_t dt \tag{2}$$

[1] Units of domestic per foreign currency, so that **XXX** is the base and **YYY** is the numeraire currency.

At time 0, the replication portfolio in a long position in the derivatives V, **YYY** cash-collateralized, is set up with a given quantity of the underlying asset and of **XXX** and **YYY** bank accounts such that their value equals the starting value of the contract and of the collateral account:

$$V_0 - C\mathrm{f}_0 = \alpha_0 S_0 + \beta_0 B_0 + \theta_0 B\mathrm{f}_0 \tag{3}$$

As usual we impose the self financing and replicating condition to find the quantities $\{\alpha_t, \beta_t, \theta\}$. We can write the evolution of the replicating portfolio as:

$$\begin{aligned}(\alpha_t dS_t + \beta_t dB_t)\mathbf{YYY} + \theta_t dB\mathrm{f}_t \mathbf{XXX} = \\ (\alpha_t(\mu_t - y_t)S_t dt + \alpha_t \sigma_t S_t dZ_t + \beta_t r_t B_t dt + \alpha_t y_t S_t dt)\mathbf{YYY} \\ + \theta_t r\mathrm{f}_t B\mathrm{f}_t dt \mathbf{XXX}\end{aligned} \tag{4}$$

On the other hand:

$$dV_t \mathbf{YYY} - dC\mathrm{f}_t \mathbf{XXX} = \left(\mathcal{L}^{\mu-y} V_t dt + \sigma_t S_t \Delta_u dZ_t\right)\mathbf{YYY} - \left[\gamma d\left(\frac{V_t}{\mathcal{X}_t}\right) + \mathrm{cf}_t C\mathrm{f}_t dt\right]\mathbf{XXX} \tag{5}$$

Equating (4) and (5) we get:

$$\begin{aligned}\left(\mathcal{L}^{\mu-y} V_t dt + \sigma_t S_t \Delta_u dZ_t\right)\mathbf{YYY} - \left[\gamma d\left(\frac{V_t}{\mathcal{X}_t}\right) + \mathrm{cf}_t C\mathrm{f}_t dt\right]\mathbf{XXX} = \\ (\alpha_t(\mu_t - y_t)S_t dt + \alpha_t \sigma_t S_t dZ_t + \beta_t r_t B_t dt + \alpha_t y_t S_t dt)\mathbf{YYY} \\ + \theta_t r\mathrm{f}_t B\mathrm{f}_t dt \mathbf{XXX}\end{aligned} \tag{6}$$

We can determine the α and β such that the stochastic part in (6) is cancelled out:

$$\begin{aligned}\alpha_t &= \Delta_t \\ \beta_t &= \frac{V_t - \Delta_t S_t}{B_t} \\ \theta_t &= -\frac{C\mathrm{f}_t}{B\mathrm{f}_t}\end{aligned} \tag{7}$$

Substituting in (6):

$$(\mathcal{L}^{r-y} V_t dt - r_t V_t dt)\mathbf{YYY} = [\gamma d(V_t/\mathcal{X}_t) - (r\mathrm{f}_t - \mathrm{cf}_t) C\mathrm{f}_t dt]\mathbf{XXX} \tag{8}$$

We can express equation (8) in terms of **YYY** only by multiplying the second term by the FX rate $\mathcal{X}_t$ and then we have:

$$\mathcal{L}^{r-y} V_t dt = r_t V_t dt + \gamma dV_t - (r\mathrm{f}_t - \mathrm{cf}_t) C_t dt \tag{9}$$

where C is the collateral account converted in **YYY** units (we suppressed the indication of the currency to lighten the notation).

It can be shown (see Castagna [1]) that the solution to equation (9) is:

$$V_0^{C\mathrm{f}} = E^Q\left[e^{-\int_0^T r_u du} V_T\right] + E^Q\left[\int_0^T e^{-\int_0^u r_v dv}(r\mathrm{f}_u - \mathrm{cf}_u) C_u du\right] \tag{10}$$

which is the same result as we have when the collateral is posted in the **YYY**, with the only difference that the collateral account amount in **YYY** is multiplied by the difference of the risk-free rate and the collateral rate applied to the collateral amount in **XXX** units. We can also denote the second part of the formula as the Liquidity Value Adjustment, or **LVA**, that is the present value of the cost incurred to finance the collateral in **XXX** units:

$$V_0^{C\mathrm{f}} = V_0^{NC} + \mathbf{LVA}$$

Recalling that $C\mathrm{f}_t = \gamma(V_t/\mathcal{X}_t)\mathbf{XXX}$, or $C_t = \gamma V_t \mathbf{YYY}$ equation (9) has also the solution:

$$V_0^{C\mathrm{f}} = E^Q\left[e^{-\int_0^T [r_u - (r\mathrm{f}_u - c\mathrm{f}_u)\gamma]du} V_T(S^{r-y})\right] \tag{11}$$

We have added the dependency of the value of the claim on the underlying price, whose drift is indicates in the superscript. In practice we can use standard valuation formulae derived, for example, in the Black&Scholes economy simply changing the discount rate: this will be no more just the domestic **YYY**-currency risk-free rate, but there will be a correction depending on the collateralization percentage γ and on the foreign **XXX**-currency risk-free and collateral rates. A similar formula has been derived by Fujii *et al.* [2].

Remark 2.1. *The value of a contract collateralized in a currency different from the one which the pay-off is denominated in, does not depend on the FX rate $\mathcal{X}$, but only on the risk-free and the collateral rate of the currency* **XXX***, beside the risk-free rate of the currency* **YYY***.*

2.1 Pricing with Funding Rate Different from Investment Rate

Assume that the operator of the replication strategy, say a bank for which the investment and funding rates are different, due mainly to credit factors. The bank pays the funding rate r^F when financing its activity in the domestic **YYY** currency; analogously $r\mathrm{f}^F$ is the rate that it pays when financing its activity in the foreign **YYY** currency. The evolution of the domestic bank account in (1) is:

$$dB_t = \widetilde{r}_t B_t dt \tag{12}$$

where $\widetilde{r}_t = r_t \mathbf{1}_{\{\beta>0\}} + r_t^F \mathbf{1}_{\{\beta<0\}}$ and $\mathbf{1}_{\{\}}$ is the indicator function equal to 1 when the condition at the subscript is verified. The **XXX** bank account evolves as follows:

$$dB\mathrm{f}_t = \widetilde{r\mathrm{f}}_t B_t dt \tag{13}$$

The funding rate can be written as the risk-free rate plus a spread:

$$\widetilde{r}_t = r_t + s_t^F \mathbf{1}_{\{\beta<0\}} \tag{14}$$

and similarly for the foreign rate.

$$\widetilde{r\mathrm{f}}_t = r\mathrm{f}_t + s\mathrm{f}_t^F \mathbf{1}_{\{\theta<0\}} \tag{15}$$

Replacing the risk-free rates r_t and $r\mathrm{f}_t$ with $\widetilde{r}_t$ and $\widetilde{r\mathrm{f}}_t$ in equation (9), one gets:

$$\mathcal{L}^{\widetilde{r}-y} V_t dt = \widetilde{r}_t V_t dt + \gamma dV_t - (\widetilde{r\mathrm{f}}_t - c\mathrm{f}_t) C_t dt \tag{16}$$

From (16) we can easily derive the two ways to express the contingent claim's value at time 0 equivalent to formulae (10) and (11), respectively as:

$$V_0^{\mathrm{Cf}} = E^Q\left[e^{-\int_0^T \widetilde{r}_u du} V_T\right] + E^Q\left[\int_0^T e^{-\int_0^u \widetilde{r}_v dv}(\widetilde{r\mathrm{f}}_u - \mathrm{cf}_u)C_u du\right] \tag{17}$$

and

$$V_0^{\mathrm{Cf}} = E^Q\left[e^{-\int_0^T [\widetilde{r}_u - (\widetilde{r\mathrm{f}}_u - \mathrm{cf}_u)\gamma]du} V_T(S^{\widetilde{r}-y})\right] \tag{18}$$

Equation (17) offers the decomposition of the collateralized contract value as the sum of the otherwise identical non-collateralized deal and of the **LVA**.

We would like also to isolate the effect due to the funding spread, so we introduce a a further decomposition by rewriting equation (16) as:

$$\mathcal{L}^{r-y}V_t dt = r_t V_t dt + \gamma dV_t - (r\mathrm{f}_t - \mathrm{cf}_t)C_t dt + s_t^F \mathbf{1}_{\{\beta<0\}}(V_t - \Delta_t S_t)dt - s\mathrm{f}_t^F \mathbf{1}_{\{\theta<0\}} C_t dt \tag{19}$$

The solution to (19) is:

$$V_0^{\mathrm{Cf}} = V^{NC} + \mathbf{LVA} + \mathbf{FVA} \tag{20}$$

where V^{NC} is the price of the non-collateralized contract assuming no funding spread, the **LVA** is the liquidity value adjustment originated by the difference between the collateral and risk-free rate:

$$\mathbf{LVA} = E^Q\left[\int_0^T e^{-\int_0^u r_v dv}(r\mathrm{f}_u - \mathrm{cf}_u)C_u du\right] \tag{21}$$

and finally **FVA** is the funding value adjustment due to the funding spread and paid to replicate the contract and the collateral account:

$$\mathbf{FVA} = E^Q\left[-\int_0^T e^{-\int_0^u r_v dv}[s_u^F \mathbf{1}_{\{\beta<0\}}(V_u - \Delta_u S_u) - s\mathrm{f}_u^F \mathbf{1}_{\{\theta<0\}} C_u]du\right] \tag{22}$$

where β has been defined above. The **FVA** is the correction to the risk-free value of the non-collateralized contract that has to be (algebraically) added to the **LVA** correction. For a definition of the **LVA** and **FVA**, see Castagna [1].

We can now analyse five different cases:

1. Assume one has to replicate a contingent claim with a constant positive sign NPV (*e.g.*: a long European call option) with a constant positive sign Δ_t. In this case $\beta_t = V_t - \Delta S_t < 0$ and $\theta_t = -C_t < 0$ always (implying borrowing in both currencies at any time). The pricing equation (19) reads then:

$$\mathcal{L}^{r-y}V_t dt = r_t V_t dt + \gamma dV_t - (r\mathrm{f}_t - \mathrm{cf}_t)C_t dt + s_t^F (V_t - \Delta_t S_t)dt - s\mathrm{f}_t^F C_t dt \tag{23}$$

   Although the decomposition in (20) still applies, the pricing can be performed very simply by means of an effective discount rate:

$$V_0^{\mathrm{Cf}} = E^Q\left[e^{-\int_0^T [r_u^F - (r\mathrm{f}_u^F - \mathrm{cf}_u)\gamma]du} V_T(S^{r^F - y})\right] \tag{24}$$

   So we can simply replace the risk-free rate with the funding rate paid by the bank and perform the same pricing as in the case when lending and borrowing rates are the equal.

2. When the same contingent claim (constant positive NPV and Δ) as in the point above is sold, the underlying asset has to be sold as well in the replication strategy, which implies that $\beta_t > 0$ and that the bank has to invest at the risk-free rate at any time in the **YYY** currency; the bank account in **XXX** currency, $\theta_t = -C_t$, will be always positive as well. The pricing formula will the same as in formula (11) (with reversed signs since we are selling the contract). In this case the **FVA** will be nil. An example of this claim is a short European call option.

3. Assume now that the contingent claim has a constant positive sign NPV, but its replication implies a negative position in the underlying asset (*e.g.*: a long European put option), then we have again that $\beta_t = V_t - \Delta S_t 0 > 0$ at any time, but $\theta_t = -C_t < 0$ at any time, implying that the bank has to borrow money in **XXX** currency. The PDE (19) reads now:

$$\mathcal{L}^{r-y} V_t dt = r_t V_t dt + \gamma dV_t - (r\mathrm{f}_t^F - \mathrm{cf}_t) C_t dt - s\mathrm{f}_t^F C_t dt \tag{25}$$

   The pricing can be performed via the compact formula:

$$V_0^{C\mathrm{f}} = E^Q \left[e^{-\int_0^T [r_u - (r\mathrm{f}_u^F - \mathrm{cf}_u)\gamma] du} V_T(S^{r-y}) \right] \tag{26}$$

   In this case we replace the risk-free rate with the funding rate paid by the bank only for the **XXX** currency.

4. If the NPV has a constant negative sign and the replica entails a long position in the underlying (*e.g.*: short European put option), then the total amount of the bank account $\beta_t = V_t - \Delta S_t < 0$ is always negative, implying that at any time the bank has to borrow money in the replica at the rate r^F in **YYY**; since $\theta_t = -C_t > 0$ at any time, the banks will invest at the risk-free rate $r\mathrm{f}$ the collateral. The pricing formula is derived similarly to (26) and it is:

$$V_0^{C\mathrm{f}} = E^Q \left[e^{-\int_0^T [r_u^F - (r\mathrm{f}_u - \mathrm{cf}_u)\gamma] du} V_T(S^{r^F - y}) \right] \tag{27}$$

5. Finally, if the NPV has a constant positive or negative sign and the Δ can flip from one sign to the other, then it is not possible to determine the sign of the bank account amounts β_t throughout the entire life of the contract, although it is always possible to determine whether θ_t is always positive or negative. In this case the pricing formula (19) cannot be reduced to a convenient representation as in the cases above, and it has to be numerically.

2.2 Funding Rate Different from Investment Rate and Repo Rate

Repo transaction should be the proper way to finance the buying of the underlying asset in the replication strategy. On the other hand, if we really want to consider the actual alternatives that a trader has to invest received sums in the less credit-risky way, reverse repo seems an effective option is most of cases. The amount to be lent/borrowed via the domestic and foreign bank accounts is now:

$$\beta_t = \frac{V_t}{B_t}$$

$$\theta_t = -\frac{C\mathrm{f}_t}{B\mathrm{f}_t}$$

whereas the quantity $\alpha_t = \Delta_t$ of underlying asset is repoed/reverse repoed, thus paying/receiving the interest $r^E_t \Delta_t S_t$. Replacing these quantities in equation (9), one gets:

$$\mathcal{L}^{r^E-y} V_t dt = \widetilde{r}_t V_t dt + \gamma dV_t - (r\mathrm{f}_t - c\mathrm{f}_t) C_t dt + s^F_t \mathbf{1}_{\{\beta<0\}} V_t dt - s\mathrm{f}^F_t \mathbf{1}_{\{\theta<0\}} C_t dt \tag{28}$$

The solution to (28) is:

$$V_0^{C\mathrm{f}} = V^{NC} + \mathbf{LVA} + \mathbf{FVA} \tag{29}$$

where, as usual, V^{NC} is the price of the non-collateralized contract assuming no funding spread and repo, the **LVA** is the liquidity value adjustment due to the collateral agreement:

$$\mathbf{LVA} = E^Q\left[\int_0^T e^{-\int_0^u r_v dv} (r\mathrm{f}_u - c\mathrm{f}_u) C_u du\right]$$

and **FVA** is the funding value adjustment:

$$\mathbf{FVA} = E^Q\left[-\int_0^T e^{-\int_0^u r_v dv} [s^F_u \mathbf{1}_{\{\beta<0\}} V_u - s\mathrm{f}^F_u \mathbf{1}_{\{\theta<0\}} C_u - s^E_t \Delta_u S_u] du\right] \tag{30}$$

The **FVA** is in this case split in the funding cost needed to finance the collateral ($s^F_u \mathbf{1}_{\{\beta<0\}} V_u - s\mathrm{f}^F_u \mathbf{1}_{\{\theta<0\}} C_u$) and the spread of repo rate over the risk-free rate ($s^E_t = r^E_t - r_t$) paid on the position of amount Δ_t of the underlying asset.

To better understand how the total **FVA** is built up, we split formula (30) in two components: the first one is $\mathbf{FVA^P}$, the cost borne to fund the premium and the collateral and it is the same as in (52). The second part refers to the repo cost to buy or to sell the underlying asset to replicate the pay-off:

$$\mathbf{FVA^R} = E^Q\left[\int_0^T e^{-\int_0^u r_v dv} s^E_t \Delta_u S_u du\right] \tag{31}$$

Also in this case it is possible to re-write (29) in a more convenient fashion for computational purposes:

$$V_0^{C\mathrm{f}} = E^Q\left[e^{-\int_0^T [\widetilde{r}_u - (\widetilde{r\mathrm{f}}_u - c\mathrm{f}_u)\gamma] du} V_T(S^{r^E-y})\right] \tag{32}$$

Formula (32) applies in the five cases analysed in the previous section: the discount factor depends on the sign of the bank account needed to fund the collateral account, whereas the drift of the underlying asset is any case the repo rate r^E.

3 FX Derivatives

We want to compute the value to the bank of an FX derivative contract: it is a function of the FX rate $\mathcal{X}$ and of time, $V(\mathcal{X}_t, t)$. we start we the the simple forward contract, named outright in the FX market.

3.1 Collateral Posted in Numeraire Currency

If the collateral is posted in the numeraire currency **YYY**, we fall in the case examined in Castagna [1] for a general derivative contract, although here we need to replace the underling asset with the exchange rate. We focus only on the more realistic case of different borrowing/lending rates and we apply as before the replication argument.

The difference between an FX and other securities' (say, equity) trade is that one is actually buying some currency money paying a price in another currency, and money received can be invested in some bank account that we assume to be default risk-free.[2] So by buying, as an example, the foreign (that we assumed to be the base) currency **XXX** the bank can invest this amount in a **XXX** denominated bank account. On the other hand, when the bank needs to short the base currency to buy numeraire currency, it has to borrow money in **XXX** currency.

The evolution of a contract $V(\mathcal{X}_t, t) = V_t$, by Ito's lemma, is:

$$dV_t = \mathcal{M}^{\mu} V_t + \sigma_t \mathcal{X}_t \frac{\partial V_t}{\partial \mathcal{X}_t} dW_t \tag{33}$$

where

$$\mathcal{M}^{a} \cdot = \frac{\partial \cdot}{\partial t} + a_t \mathcal{X}_t \frac{\partial \cdot}{\partial \mathcal{X}_t} + \frac{1}{2} \nu_t^2 \mathcal{X}_t^2 \frac{\partial^2 \cdot}{\partial \mathcal{X}_t^2} \tag{34}$$

The replicating portfolio comprises, at time t, a given amount α_t of the base currency **XXX**, worth $\mathcal{X}_t$, and a given amount of cash β_t, borrowed or invested, in currency **YYY**. The portfolio must equal the value of the FX derivative at time 0:

$$V_0 - C_0 = \alpha_0 \mathcal{X}_0 + \beta_0 B_0 \tag{35}$$

Considering the fact that the α units of **XXX** are either invested or borrowed in a bank account, depending on the sign of α, the evolution of the replicating portfolio is:

$$\begin{aligned} &(\alpha_t d\mathcal{X}_t + \beta_t dB_t)\mathbf{YYY} + \alpha_t dB\mathrm{f}_t \mathbf{XXX} = \\ &(\alpha_t d\mathcal{X}_t + \beta_t dB_t + \alpha_t \mathcal{X}_t dB\mathrm{f}_t)\mathbf{YYY} = \\ &\quad \alpha_t \eta_t \mathcal{X}_t dt + \alpha_t \nu_t \mathcal{X}_t dZ_t + \beta_t \widetilde{r}_t B_t dt + \alpha_t \widetilde{r\mathrm{f}}_t \mathcal{X}_t dt \end{aligned} \tag{36}$$

where $\widetilde{r\mathrm{f}}_t = r\mathrm{f}_t + s\mathrm{f}_t^F \mathbf{1}_{\{\alpha<0\}}$ and the denomination in the **YYY** currency has been omitted in the last line. Setting $\alpha_t = \Delta_t = \frac{\partial V_t}{\partial \mathcal{X}_t}$ and $\beta_t = (V_t - C_t - \Delta_t \mathcal{X}_t)/B_t$, and following the same passages as in Castagna [1], we come up with the PDE:

$$\mathcal{L}^{\widetilde{r} - \widetilde{r\mathrm{f}}} V_t dt = \widetilde{r}_t V_t dt + \gamma dV_t - (\widetilde{r}_t - c_t) C_t dt \tag{37}$$

[2] In practice the bank will never be able to find a completely risk-free counterparty, but if the interest yielded by the bank account issued by the latter is fair, it should include the remuneration for the expected default losses, so that on a risk-adjusted basis the net yield is still the risk-free rate.

From (37) we can easily derive the two ways to express the contingent claim's value at time 0:

$$V_0^C = E^Q\left[e^{-\int_0^T \widetilde{r}_u du} V_T(\mathcal{X}^{\widetilde{r}-\widetilde{r\mathrm{f}}})\right] + E^Q\left[\int_0^T e^{-\int_0^u \widetilde{r}_v dv}(\widetilde{r}_u - c_u)C_u du\right] \tag{38}$$

and

$$V_0^C = E^Q\left[e^{-\int_0^T [\widetilde{r}_u(1-\gamma)+c_u\gamma]du} V_T(\mathcal{X}^{\widetilde{r}-\widetilde{r\mathrm{f}}})\right] \tag{39}$$

We have explicitly indicated which is the drift that the FX rate must have under the bank's replication measure. Equation (38) offers the decomposition of the collateralized contract value as the sum of the otherwise identical non-collateralized deal and of the **LVA**.

We introduce a further decomposition that can be useful to allocate revenues and costs within a dealing room, we rewrite equation (37) as:

$$\mathcal{L}^{r-r\mathrm{f}} V_t dt = r_t V_t dt + \gamma dV_t - (r_t - c_t)C_t dt + s_t^F \mathbf{1}_{\{\beta<0\}}(V_t - C_t - \Delta_t \mathcal{X}_t)dt + s\mathrm{f}_t^F \mathbf{1}_{\{\alpha<0\}}\Delta_t \mathcal{X}_t dt \tag{40}$$

The solution to (40) is:

$$V_0^C = V^{NC} + \mathbf{LVA} + \mathbf{FVA} \tag{41}$$

where V^{NC} is the price of the non-collateralized contract assuming no funding spread, the **LVA** is the liquidity value adjustment originated by the difference between the collateral and risk-free rate:

$$\mathbf{LVA} = E^Q\left[\int_0^T e^{-\int_0^u r_v dv}(r_u - c_u)C_u du\right] \tag{42}$$

and finally **FVA** is the funding value adjustment due to the funding spread and paid to replicate the contract and the collateral account:

$$\mathbf{FVA} = E^Q\left[-\int_0^T e^{-\int_0^u r_v dv}[s_t^F \mathbf{1}_{\{\beta<0\}}(V_t - C_t - \Delta_t \mathcal{X}_t) + s\mathrm{f}_u^F \mathbf{1}_{\{\alpha<0\}}\Delta_u \mathcal{X}_u du]\right] \tag{43}$$

The **FVA** can be decomposed in the funding adjustment due to the spread paid in the **YYY** currency:

$$\mathbf{FVA}^{\mathbf{YYY}} = E^Q\left[-\int_0^T e^{-\int_0^u r_v dv} s_t^F \mathbf{1}_{\{\beta<0\}}(V_t - C_t - \Delta_t \mathcal{X}_t)du\right] \tag{44}$$

and the funding adjustment due to the spread paid in **XXX** currency:

$$\mathbf{FVA}^{\mathbf{XXX}} = E^Q\left[-\int_0^T e^{-\int_0^u r_v dv} s\mathrm{f}_u^F \mathbf{1}_{\{\alpha<0\}}\Delta_u \mathcal{X}_u du\right] \tag{45}$$

3.2 Collateral Posted in Base Currency

When collateral is posted in the base currency **XXX** we can apply the results we have derived above for a general derivative contract to the case of a an FX derivative contract. The replicating portfolio is built as follows

$$V_0 - C_0 = \alpha_0 \mathcal{X}_0 + \beta_0 B_0 + \theta_0 B\mathrm{f} \tag{46}$$

and its evolution is:

$$\alpha_t\eta_t\mathcal{X}_t dt + \alpha_t\nu_t\mathcal{X}_t dZ_t + \beta_t\widetilde{r}_t B_t dt + \theta\widetilde{r\mathrm{f}}_t B\mathrm{f}_t dt + \alpha_t\widetilde{r\mathrm{f}}_t\mathcal{X}_t dt \tag{47}$$

Choosing $\beta_t = (V_t - \Delta_t\mathcal{X}_t)/B_t$ and $\theta = -(C\mathrm{f}_t - \Delta_t)/B\mathrm{f}_t$**XXX** $= -(C_t - \Delta_t\mathcal{X}_t)/B\mathrm{f}_t$ **YYY**, we derive the following PDE:

$$\begin{aligned}\mathcal{L}^{r-r\mathrm{f}}V_t dt =& r_t V_t dt + \gamma dV_t \\ &- (r\mathrm{f}_t - c\mathrm{f}_t)C_t dt + s_t^F \mathbf{1}_{\{\beta<0\}}(V_t - \Delta_t\mathcal{X}_t)dt - s\mathrm{f}_t^F \mathbf{1}_{\{\theta<0\}}(C_t - \Delta_t\mathcal{X}_t)dt\end{aligned} \tag{48}$$

The solution is

$$V_0^{C\mathrm{f}} = E^Q\left[e^{-\int_0^T[\widetilde{r}_u - (\widetilde{r\mathrm{f}}_u - c\mathrm{f}_u)\gamma]du}V_T(\mathcal{X}^{\widetilde{r}-\widetilde{r\mathrm{f}}})\right] \tag{49}$$

The solution to (48) is also:

$$V_0^{C\mathrm{f}} = V^{NC} + \mathbf{LVA} + \mathbf{FVA} \tag{50}$$

where V^{NC} is the price of the non-collateralized contract assuming no funding spread, the **LVA** is the liquidity value adjustment originated by the difference between the collateral and risk-free rate:

$$\mathbf{LVA} = E^Q\left[\int_0^T e^{-\int_0^u r_v dv}(r\mathrm{f}_u - c\mathrm{f}_u)C_u du\right] \tag{51}$$

and finally **FVA** is the funding value adjustment due to the funding spread and paid to replicate the contract and the collateral account:

$$\mathbf{FVA} = E^Q\left[-\int_0^T e^{-\int_0^u r_v dv}[s_u^F \mathbf{1}_{\{\beta<0\}}(V_u - \Delta_u\mathcal{X}_u) - s\mathrm{f}_u^F \mathbf{1}_{\{\theta<0\}}(C_u - \Delta_u\mathcal{X}_u)]du\right] \tag{52}$$

3.3 Value of an FX Forward (Outright) Contract

Assume collateral is posted in currency **YYY**. A (long) FX forward contract, or outright, struck at level X has a terminal value:

$$V_T = \mathcal{X}_T - X \tag{53}$$

so that applying the compact formula (39)

$$V_0^C = E^Q\left[e^{-\int_0^T[\widetilde{r}_u(1-\gamma)+c_u\gamma]du}(\mathcal{X}_T - X)\right] \tag{54}$$

The value at inception of the contract is nil: if we do not consider for the moment all the adjustments due to the default risk of the bank and of its counterparty, we can price the contract and find which is the level $X = X^C(t,T)$ that makes zero the value at the beginning of the contract, with formula (38). If the bank needs to replicate a long position in the outright contract, the the outright price can be easily shown to be:

$$X^C(t,T) = \mathcal{X}_0 e^{\int_0^T(r_u + s_u^F - r\mathrm{f}_u)du} \tag{55}$$

On the other hand, when the banks wants to replicate a short position, the outright price is:

$$X^C(t,T) = \mathcal{X}_0 e^{\int_0^T(r_u - r\mathrm{f}_u - s\mathrm{f}_u^F)du} \tag{56}$$

Remark 3.1. *In both cases the collateralization, and hence the collateral rate, are not affecting the fair level of the outright contract, although the* **LVA** *contributes to the value of the contract when the outright is seasoned and no more at-the-money as at the inception. On the contrary, the funding spreads paid on either currencies (***YYY** *or* **XXX***) are entering in the formula and they are crucial to define both the replication value of the contract to the bank and the fair level.*

Remark 3.2. *From* (40) *it is quite manifest that we are still within a risk-neutral framework, where everything is discounted with the risk-free rate and the drift of the FX rate process $\mathcal{X}_t$ is the difference between the numeraire and base currency: a standard result. Using PDE* (37) *leads to more convenient valuation formulae, but in our opinion it makes less clear how the value is composed, and why it can be different to different parties, although we are still working in a dynamic replication setting that produces a risk-neutral value.*[3]

If the collateral is posted in **XXX** currency, then the forward price is the level making zero the contract's value at inception computed via the PDE (48) whose solution can be written as the compact formula (49), so that

$$V_0^{C\mathrm{f}} = E^Q\left[e^{-\int_0^T [\widetilde{r}_u - (\widetilde{r\mathrm{f}}_u - c\mathrm{f}_u)\gamma] du}(\mathcal{X}_T - X)\right] \tag{57}$$

It is quite easy to check that both the long and short outright fair price level is the same as in formulae (55) and (56), so that $X^C(t,T) = X^{C\mathrm{f}}(t,T)$, with $X^{C\mathrm{f}}$ the outright fair price at time t for a maturity T when the collateral is posted in base currency.

Remark 3.3. *Although the fair level of the outright contract is independent from the currency which the collateral is posted in, the value of the contract do depend on it. The values of two contracts collateralized one in the numeraire and the other in the base currency differ during their life, being equal only at the inception (i.e.: zero) and at the expiry.*

3.4 Replication with FX Swap

The funding spread in both currencies can be strongly abated if the bank uses collateralized instead of unsecured lending. In the FX this can be easily achieved via an FX swap, that is in all respect equal to a repo traded in other markets. The FX swap is the sum of a spot contract plus an outright, but it can also be seen as the borrowing/lending of an **XXX** amount against collateral represented by the **YYY** amount.[4]

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[3]Just the value, NOT the price, is risk-neutral. This means that an economic agent bearing the same costs to replicate the contract, agrees on the value of the contract, independently from its risk aversion. The replication, in presence of funding and collateral costs, depends on the long or short position one wishes to reproduce.

[4]It should be noted that the borrowing is collateralized in a static fashion at the start of the contract (*i.e.*: the amount of the currency one of the party pays against receiving the amount of the other currency).

FX swaps for a given expiry T are quoted in the market in points over the spot rate $\mathcal{X}$, so that the level at which the outright is traded is defined as $\mathcal{F}(t,T) = \mathcal{X}_t + p(t,T)$, where $p(t,T)$ are the swap points prevailing at time t for an FX swap expiring at time T. The outright level defines also the FX swap implied rate, that depends mainly on the differential between the numeraire and base currency, but also on other factors (even beyond credit risk) that determine the generally defined cross-currency basis. The implied FX swap (continuous) rate is defined as:

$$r_t^{\mathcal{X}} = -\frac{\partial \ln \frac{\mathcal{F}(t,T)}{\mathcal{X}_t}}{\partial t} \tag{58}$$

Operating with an FX swap to replicate the FX derivative contract and assuming for the moment that the FX swap is CSA-collateralized in **YYY** currency, we have that formula (36) modifies as follows:

$$\alpha_t \eta_t \mathcal{X}_t dt + \alpha_t \nu_t \mathcal{X}_t dZ_t + \beta_t \widetilde{r}_t B_t dt - \alpha_t r_t^{\mathcal{X}} \mathcal{X}_t dt \tag{59}$$

Setting $\alpha_t = \Delta_t = \frac{\partial V_t}{\partial \mathcal{X}_t}$ and $\beta_t = (V_t - C_t)$, we have that the evaluation PDE is:

$$\mathcal{L}^{r^{\mathcal{X}}} V_t dt = \widetilde{r}_t V_t dt + \gamma dV_t - (\widetilde{r}_t - c_t) C_t dt \tag{60}$$

The solution to (60) is:

$$V_0^C = V^{NC} + \mathbf{LVA} + \mathbf{FVA} \tag{61}$$

where, again, V^{NC} is the price of the non-collateralized contract on the exchange rate §, assuming no funding spread and repo rate; the **LVA** is the liquidity value adjustment due to the collateral agreement:

$$\mathbf{LVA} = E^Q\left[\int_0^T e^{-\int_0^u r_v dv}(r_u - c_u) C_u du\right]$$

and **FVA** is the funding value adjustment:

$$\mathbf{FVA} = E^Q\left[-\int_0^T e^{-\int_0^u r_v dv}[s_u^F \mathbf{1}_{\{\beta<0\}}(V_u - C_u) - s_t^{\mathcal{X}} \Delta_u \mathcal{X}_u] du\right] \tag{62}$$

The **FVA** is also in this split in the funding cost needed to finance the collateral ($s_u^F \mathbf{1}_{\{\beta<0\}} V_u - sf_u^F \mathbf{1}_{\{\theta<0\}} C_u$) and the spread of repo rate over the risk-free rate ($s_t^{\mathcal{X}} = r_t^{\mathcal{X}} - r_t$) paid on the position of amount Δ_t of the underlying asset.

It is possible to re-write (68) in a more convenient fashion for computational purposes:

$$V_0^C = E^Q\left[e^{-\int_0^T [\widetilde{r}_u(1-\gamma) + c_u \gamma] du} V_T(\mathcal{X}^{r^{\mathcal{X}}})\right] \tag{63}$$

This static collateral is not daily readjusted as in the case of the CSA agreement, so there is still the risk that, upon counterparty's default, the market value is not able to full protect the loss suffered by the surviving party.

The FX swap can be used to replicate the outright contract we have shown above: in the end the FX swap is just the replication strategy of an outright operated with a single counterparty, thus minimizing the losses given default and hence the spread paid. When considering also the fact that the FX swap, being a derivative contract itself, is CSA-collateralized, we also get the same cash-flows' profile for both the outright and the FX swap, so that funding spreads should not be considered in the evaluation process. The replica of the contract is then independent from the creditworthiness of the replicator bank. This means the in practice they are the same contract when CSA agreement is operating and that the outright fair price is just the FX swap price:

$$X^C = \mathcal{F}(t,T) = \mathcal{X}_0 e^{\int_0^T r_u^{\mathcal{X}} du} \tag{64}$$

The dynamics for the FX rate, when replication is operated via the FX swap, is:

$$d\mathcal{X}_t = r_t^{\mathcal{X}} \mathcal{X}_t dt + \nu_t \mathcal{X}_t dW_t \tag{65}$$

Let us see now what happens if the replication is performed with a repo contract and the collateral is posted in the base currency (**XXX**). Equation (47) modifies as follows:

$$\alpha_t \eta_t \mathcal{X}_t dt + \alpha_t \nu_t \mathcal{X}_t dZ_t + \beta_t \widetilde{r}_t B_t dt + \theta \widetilde{r\mathrm{f}}_t B\mathrm{f}_t dt + \alpha_t r_t^{\mathcal{X}} \mathcal{X}_t dt \tag{66}$$

Setting $\beta_t = V_t/B_t$ and $\theta = -C\mathrm{f}_t/B\mathrm{f}_t \mathbf{XXX} = -C_t/B\mathrm{f}_t\ \mathbf{YYY}$, we derive the following PDE:

$$\mathcal{L}^{r^{\mathcal{X}}} V_t dt = \widetilde{r}_t V_t dt + \gamma dV_t - (\widetilde{r\mathrm{f}}_t - c\mathrm{f}_t) C_t dt \tag{67}$$

where $\widetilde{r\mathrm{f}}_t = r\mathrm{f}_t + s\mathrm{f}_t^F \mathbf{1}_{\{\theta<0\}}$. The solution to (67) is:

$$V_0^{C\mathrm{f}} = V^{NC} + \mathbf{LVA} + \mathbf{FVA} \tag{68}$$

where, as usual, V^{NC} is the price of the non-collateralized contract on the exchange rate §, assuming no funding spread and repo rate; the **LVA** is the liquidity value adjustment due to the collateral agreement:

$$\mathbf{LVA} = E^Q\left[\int_0^T e^{-\int_0^u r_v dv}(r\mathrm{f}_u - c\mathrm{f}_u) C_u du\right]$$

and **FVA** is the funding value adjustment:

$$\mathbf{FVA} = E^Q\left[-\int_0^T e^{-\int_0^u r_v dv}[s_u^F \mathbf{1}_{\{\beta<0\}} V_u - s\mathrm{f}_u^F \mathbf{1}_{\{\theta<0\}} C_u - s_t^{\mathcal{X}} \Delta_u \mathcal{X}_u] du\right] \tag{69}$$

with $s_t^{\mathcal{X}} = r_t^{\mathcal{X}} - r_t$ as above. The compact solution in this case is:

$$V_0^{C\mathrm{f}} = E^Q\left[e^{-\int_0^T [\widetilde{r}_u - (\widetilde{r\mathrm{f}}_u - c\mathrm{f}_u)\gamma] du} V_T(\mathcal{X}^{r^{\mathcal{X}}})\right] \tag{70}$$

It is straightforward to check that also when the FX swap is used to replicate a forward contract, the currency of the collateral is immaterial, since from (70) one can derive the fair level outright level when the collateral is posted in the base currency, which is the same as in (64), so that: $X^{C\mathrm{f}} = X^C$.

4 Interest Rate Derivatives

We argued in Castagna [1] that interest rate derivatives should be considered as primary securities, so that pricing formula cannot be derived by a true replication argument, but they are simply market pricing formulae. We illustrate how to evaluate two basic contracts of the interest derivatives market, that is the the Forward Rate Agreement (**FRA**) and an Interest Rate Swap (**IRS**).

4.1 Forward Rate Agreement

Assume we have the risk-free discount curve in both currencies denoted by D and D^f, respectively, for the **YYY** and **XXX** currencies. The pricing formula for a **FRA** written on the a Libor rate $L_i(T_{i-1})$ of the **YYY**, but with a collateral posted in the **XXX** can be written in a fashion resembling the one presented above for contracts on other underlying assets and derived from a replication argument:

$$\mathbf{FRA}^{C\mathrm{f}}(t;T_{i-1},T_i) = P^D(t,T_i)\tau_i E_D^{T_i}[L_i(T_{i-1}) - K] + \mathbf{LVA}_{\mathbf{FRA}^{C\mathrm{f}}(t;T_{i-1},T_i)} \tag{71}$$

that is the expected Libor rate under the T_i-forward measure of the value of the contract at the expiry T_{i-1}, plus the **LVA**. We used $\tau_i = T_i - T_{i-1}$.

The **LVA** is the present value of the difference between the risk-free rate $L_j^{D^f}(t)$ and the collateral rate $O_j^f(t)$, fixed in date t_{j-1}, and valid until date t_j, both for the currency **XXX**, applied to fraction γ of the value of the contract $\mathbf{FRA}(t_j;T_{i-1},T_2)$ for all the N days between t and the forward settlement T_1, so that $t_N = T_1$:

$$\mathbf{LVA}_{\mathbf{FRA}^{C\mathrm{f}}(t;T_{i-1},T_i)} = \sum_{j=1}^{N} P^D(t,t_j)E_D^{t_j}\left[\tau_j^C[L_j^{D^f}(t) - O_j^f(t)]\gamma\mathbf{FRA}^{C\mathrm{f}}(t_j;T_{i-1},T_i)\right] \tag{72}$$

where $\tau_j^C = t_j - t_{j-1}$ is the difference in year fraction between two rebalancing times of the collateral, (typically one day). We can assume that the market quotes for **FRA**'s refers to the case when **LVA** is nil, implying that the collateral rate is supposed to be the risk-free rate $L^{D^f}(t;t_{j-1},t_j) = O^f(t;t_{j-1},t_j)$, for all j: this is not unreasonable given that standard CSA between banks provides for a remuneration of the collateral account at the OIS and the latter can be considered a very good proxy for the risk-free rate. Equation (71) will be then:

$$\mathbf{FRA}^{C\mathrm{f}}(t;T_{i-1},T_i) = P^D(t,T_i)\tau_i E_D^{T_i}[L_i(T_{i-1}) - K] \tag{73}$$

which is exactly the same result of Castagna [1] and of the current pricing theory based on a multi-curve set-up (see for example Mercurio [3]). The **FRA** fair rate is the expected value of the Libor at the settlement date of the contract, under the expiry T_i-forward risk measure:

$$K^{C\mathrm{f}} = L_i(t) = E_D^{T_i}[L_i(T_{i-1})] \tag{74}$$

Since according to market conventions the contract actually settles the present value of the pay-off in T_i in T_{i-1}, **FRA** fair rate in (74), since the latter should be corrected by a convexity adjustment as discussed in Mercurio [4]. The adjustment is nevertheless quite small (a fraction of a basis point).

Assume now that collateral agreement provides for a remuneration of the collateral different from the OIS rate: setting $Q_i^f(t) = L_i^{Df}(t) - O_i^f(t)$ equal to the spread between the daily risk-free rate and collateral rate in **XXX** currency, and assuming it is a stochastic process independent from the value of the **FRA**, we can rewrite equation (72) as:

$$\mathbf{LVA}_{\mathbf{FRA}^{C\mathrm{f}}(t;T_{i-1},T_i)} = \sum_{j=1}^{N} P^D(t,t_j) E_D^{t_j}[\tau_j^C Q_j^f(t)] E_D^{t_j}[\gamma \mathbf{FRA}^{C\mathrm{f}}(t_j;T_{i-1},T_i)] \tag{75}$$

The second expectation in (75) is $P^D(t,T_i)E_D^{T_i}[\gamma(L_i(T_{i-1}) - K)]/P^D(t,t_j)$, so that we finally get:

$$\mathbf{LVA}^{C\mathrm{f}}_{\mathbf{FRA}(t;T_{i-1},T_i)} = \sum_{j=1}^{N} P^D(t,t_j) \left[E_D^{t_j}[\tau_j^C Q_j^f(t)] \frac{P^D(t,T_i)E_D^{T_i}[\gamma(L_i(T_{i-1}) - K)]}{P^D(t,t_j)} \right] \tag{76}$$

Apart from the slight change of notation, the results are the same as in Castagna [1] and this happens also when we consider the **FVA**. To this end, define $L^{Ff}(t;t_{i-1},t_i) = L_i^{Ff}(t)$ as the funding rate paid by the bank in the currency **XXX**. When financing the collateral (*i.e.*: when the NPV of the contract is negative to the bank) it has to pay this rate and receive the collateral rate, whereas in the opposite situation (*i.e.*: positive NPV), then it invests at the risk-free rate the collateral received, paying the collateral rate. Let $U^f(t;t_{j-1},t_j) = U_j^f(t) = L_j^{Ff}(t)$- $L_j^{Df}(t)$ be the funding spread over the risk-free rate: we assume also in this case that it is not correlated with the NPV of the **FRA**. The **FVA** is:

$$\mathbf{FVA}_{\mathbf{FRA}^{C\mathrm{f}}(t;T_{i-1},T_i)}(t;T_{i-1},T_i) = \sum_{j=1}^{N} P^D(t,t_j) \left[E_D^{t_j}[\tau_j^C U_j^f(t)] \frac{P^D(t,T_i)E_D^{T_i}[\gamma(L_i(T_{i-1}) - K)^-]}{P^D(t,t_j)} \right] \tag{77}$$

where $E[X^-] = E[\min(X,0)]$. It is straightforward to show that:

$$\frac{P^D(t,T_i)E_D^{T_i}[\gamma(L_i(T_{i-1}) - K)^-]}{P^D(t,t_j)} = -\frac{[\gamma\tau_i \mathbf{Floorlet}(t_j;T_{i-1},T_i,K)]}{P^D(t,t_j)}$$

where $\mathbf{Floorlet}(t_j;T_{i-1},T_i,K)$ is the price of a floorlet priced at time t_j, expiry in T_{i-1}, settlement in T_i, and strike K. If the bank has a short position in the **FRA**, then the **FVA** is

$$\frac{P^D(t,T_i)E_D^{T_i}[\gamma(K - L_i(T_{i-1}))^-]}{P^D(t,t_j)} = -\frac{[\gamma\tau_i \mathbf{Caplet}(t_j;T_{i-1},T_i,K)]}{P^D(t,t_j)}$$

where $\mathbf{Caplet}(t_i;T_{i-1},T_i,K)$ is the price of a caplet, and the arguments of the function are the same as for the floorlet. The total value of the **FRA** is:

$$\mathbf{FRA}^{C\mathrm{f}}(t;T_{i-1},T_i) = P^D(t,T_i)\tau_i E_D^{T_i}[L_i(T_{i-1}) - K] + \mathbf{LVA}_{\mathbf{FRA}^{C\mathrm{f}}(t;T_{i-1},T_i)} + \mathbf{FVA}_{\mathbf{FRA}^{C\mathrm{f}}(t;T_{i-1},T_i)} \tag{78}$$

The fair rate making zero the value of the contract at inception, has to be computed recursively.

Remark 4.1. *Formula* (73) *shows that the* **FRA** *fair rate is independent from the currency which the collateral is posted in: this is the same result we have derived above for derivatives with different underlying assets that can be replicated by a dynamic strategy. A difference between the* **FRA** *fair rates, refreeing to contracts collateralized in two different currencies, can be caused by a more difficult access to the money market in the* **XXX** *currency for the banks operating in the domestic country (where the* **YYY** *currency is used). This will produce a generalized and higher funding costs generally borne by domestic banks when posting collateral in the foreign* **XXX** *currency, so that an average* **FVA**, *which is typically paid by all banks, is added to* **XXX**-*collateralized* **FRA**. *In this case the* **FRA** *fair rate will be dependent on the currency chosen for the collateral to post.*

4.2 Interest Rate Swap

We will rapidly present the results for an **IRS**: they are derived by the same token as those for an **FRA**, and in any case they are the same as in Castagna [1], whom we refer to for more details.

Consider an **IRS** whose fixed leg pays a rate is K on dates $T_c^S, ..., T_d^S$ ($\tau_k^S = T_i^S - T_{i-1}^S$); the floating leg receives the Libor fixings on dates $T_a, ..., T_b$. We assume that the set of floating rate dates include the set of fixed rate dates. For both legs the present value of these payments is obtained by discounting them with the **YYY** discount curve D. If the collateral is posted in **XXX** currency, the value at time t of the **IRS** is:

$$\mathbf{IRS}^{C\mathrm{f}}(t, K; T_a, ..., T_b, T_c^S, ..., T_c^S) = \left[\sum_{k=a}^{b} P^D(t, T_k)\tau_k L_k(t) - \sum_{j=c}^{d} P^D(t, T_j)\tau_j^S K\right] + \mathbf{LVA}_{\mathbf{IRS}^{C\mathrm{f}}(t;T_a,T_b)} \tag{79}$$

with:

$$\mathbf{LVA}_{\mathbf{IRS}^{C\mathrm{f}}(t;T_a,T_b)} = \sum_{j=1}^{N} P^D(t, t_j) E_D^{t_j}\left[\tau_j^C [L_j^{D\mathrm{f}}(t) - O_j^f(t)]\gamma \mathbf{IRS}^{C\mathrm{f}}(t_j; T_a, T_b)\right] \tag{80}$$

where $\mathbf{IRS}^{C\mathrm{f}}(t; T_a, T_b) = \mathbf{IRS}^{C\mathrm{f}}(t, K; T_a, ..., T_b, T_c^S, ..., T_c^S)$. The **LVA** is as usual the difference between the **XXX** currency risk-free rate and the collateral rate applied to the fraction γ of the NPV, for all the N days occurring between the valuation date t and the end of the contract $t_N = T_b$. Also for swaps, we can make the assumption that the market quotes refer to the situation when the **LVA** $= 0$, implying that the **XXX** currency risk-free and collateral rates are the same. When the two rates are different, the **LVA** is:

$$\mathbf{LVA}_{\mathbf{IRS}^{C\mathrm{f}}(t;T_a,T_b)} = \sum_{j=1}^{N} P^D(t, t_j) E_D^{t_j}[\tau_j^C Q_j^f(t)] E_D^{t_j}[\gamma \mathbf{IRS}^{C\mathrm{f}}(t_j; T_a, T_b)] \tag{81}$$

The second expectation in (75) is $C_D^{a,b}(t) E_D^{a,b}[\gamma(S_{a,b}(t) - K)]/P^D(t, t_j)$, where $E_D^{a,b}$ is the expectation taken under the swap measure, with numeraire equal to the annuity $C_D^{a,b}(t) = \sum_{j=a+1}^{b} P^D(t, T_j)\tau_j^S$. So we can write:

$$\mathbf{LVA}_{\mathbf{IRS}^{C\mathrm{f}}(t;T_a,T_b)} = \sum_{j=1}^{N} P^D(t, t_j)\left[E_D^{t_j}[\tau_j^C Q_j^f(t)] \frac{C_D^{a,b}(t) E_D^{a,b}[\gamma(S_{a,b}(t) - K)]}{P^D(t, t_j)}\right] \tag{82}$$

The **FVA** is defined analogously to the **FRA**'s case:

$$\mathbf{FVA}_{\mathbf{IRS}^{C\mathrm{f}}(t;T_a,T_b)} = \sum_{j=1}^{N} P^D(t,t_j)\left[E_D^{t_j}[\tau_j^C U_j^f(t)]\frac{C_D^{a,b}(t)E_D^{a,b}[\gamma(S_{a,b}(t)-K)^-]}{P^D(t,t_j)}\right] \tag{83}$$

Introducing options on swaps, second expectation in (83) is:

$$\frac{C_D^{a,b}(t)E_D^{a,b}[\gamma(S_{a,b}(t)-K)^-]}{P^D(t,t_j)} = -\frac{[\gamma\mathbf{Rec}(t_j;T_a,T_b)]}{P^D(t,t_j)}$$

where $\mathbf{Rec}(t;T_a,T_b)$ is the price of a receiver swaption priced at time t_j, expiry in T_a, on a swap starting in T_a and maturing in T_b, and strike K. If the bank has a short position in the **IRS** (*i.e.*: it is a fixed rate receiver), then the **FVA** is

$$\frac{C_D^{a,b}(t)E_D^{a,b}[\gamma(K-S_{a,b}(t))^-]}{P^D(t,t_j)} = -\frac{[\gamma\mathbf{Pay}(t_j;T_a,T_b)]}{P^D(t,t_j)}$$

where $\mathbf{Pay}(t;T_a,T_b)$ is the price of a payer swaption. So the total value of the **IRS** can be written is:

$$\begin{aligned}\mathbf{IRS}^{C\mathrm{f}}(t,K;T_a,...,T_b,T_c^S,...,T_c^S) = &\left[\sum_{k=a}^{b} P^D(t,T_k)\tau_k L_k(t) - \sum_{j=c}^{d} P^D(t,T_j)\tau_j^S K\right]\\ &+ \mathbf{LVA}_{\mathbf{IRS}^{C\mathrm{f}}(t;T_a,T_b)}\\ &+ \mathbf{FVA}_{\mathbf{IRS}^{C\mathrm{f}}(t;T_a,T_b)}\end{aligned} \tag{84}$$

It is worth noticing that also fair market swap rates are independent from the currency used to post collateral, although the same considerations as those in remark apply.

5 Cross Currency Swaps

Cross currency swaps (**CCS**) involve at least two currencies since they are a periodic exchange of Libor rates in one currency against Libor rates of another currency, usually with a basis spread paid over one of them. Most of **CCS** are against US dollars, and the basis spread quoted in the market is paid over the Libor of the other currency of the deal. We analyse here how to price a **CCS** when the collateral is posted in US dollars, which we name "major currency" in what follows and that can be thought as the **XXX** we introduced above.[5] The other currency is named "minor currency" and it is the **YYY** currency. To avoid any confusion, we will add the superscript X or Y to refer to the relative currency whenever needed.

Let T_b be the expiry of a swap starting in T_a with a fixed rate $S_{a,b}(t)$. Let $L_k^X(t)$ be the forward Libor rate corresponding to the payment frequency of the floating leg (*e.g.*:

[5] This seems to be the standard when swaps are cleared via Swapclear. Although in this case collateralization process involves a multi-lateral netting, its mechanics is the same as in the bi-lateral CSA agreement and hence also the valuation of the contract can be operated in the same way.

6 month Libor for semiannual payments) for the period between T_{k-1} and T_k, computed at time t, with $T_k \geq T_a$ and $T_a \leq t < T_k$.. The notation is the same as above. $P^{D^X}(t, v)$ the US risk-free discount factor for the period t to v. If $S^X_{a,b}(t)$ is the fair market rate of the swap, the **IRS** can be computed under the assumption of nil **LVA**, so that:

$$\mathbf{IRS}^X(t, S_{a,b}; T_a, ..., T_b, T^S_c, ..., T^S_c) = \left[\sum_{k=a}^{b} P^{D^X}(t, T_k)\tau_k L^X_k(t) - \sum_{j=c}^{d} P^{D^X}(t, T_j)\tau^S_j S^X_{a,b}(t) \right] \tag{85}$$

The swap inn collateralized in **XXX** currency. The first term in the sum on the right-hand side is the present value of the stream of floating rate cash-flows, whereas the second is the present value of the fixed leg payments. The sum is nil for par swaps. For convenience in the following we indicate with $\mathbf{Float}^X(t, T_a, T_b) = \sum_{k=a}^{b} P^{D^X}(t, T_k)\tau_k L^X_k(t)$ the present value of the floating leg. We can define analogously an **IRS** in minor currency, collateralized in major currency, with fair swap rate $S^Y_{a,b}(t)$.

Let $\mathbf{CCS}^X(t, b^{ccs}_{a,b}; T_a, ..., T_b)$ be a cross currency swap against US dollar, with the same start T_a and maturity T_b and the same frequency as the standard **IRS** in (85) for both the floating legs denominated in the two currencies; $b^{ccs}_{a,b}$ is the basis paid over the Libor L^Y of the minor currency leg. The collateral is posted in US dollars (or **XXX** currrency). The value of the **YYY**-receiver **CCS**, considering also the **LVA** and the **FVA**, is:

$$\begin{aligned}\mathbf{CCS}^X(t, b^{ccs}_{a,b}; T_a, ..., T_b) = \Bigg[& \mathcal{X}_t \Bigg(\sum_{k=a}^{b} P^{D^Y}(t, T_k)\tau^Y_k \left(E^{T_k}_{D^Y}[L^Y_k(T_{k-1})] + b^{ccs}_{a,b} \right) \\ & + P^{D^Y}(t, T_a) - P^{D^Y}(t, T_b) \Bigg) \\ & - \sum_{k=a}^{b} P^{D^X}(t, T_k)\tau^X_k E^{T_i}_D[L^X_k(T_{k-1})] + P^{D^X}(t, T_a) - P^{D^X}(t, T_b) \Bigg] \\ & + \mathbf{LVA}_{\mathbf{CCS}^X} + \mathbf{FVA}_{\mathbf{CCS}^X}\end{aligned} \tag{86}$$

Let us focus on the first part of (86) in the square brackets, postponing the analysis of the $\mathbf{LVA}_{\mathbf{CCS}}$. To price a **CCS**, it is convenient to adopt the point of view of an agent operating in the major (USD) currency economy. Therefore we need to see how to evaluate a Libor payment in the minor currency when seen from the major currency economy.

To this end, assume we are given, at time $t = 0$, with i) the discount factors $P^{D^X}(0, T)$ for the major currency, ii) minor currency par swap rate paying $S^Y_{0,b}$, assuming that the swap is collateralized in the major currency, and cross-currency basis swap spreads $b^{ccs}_{0,b}$, and iii) the spot exchange rate $\mathcal{X}$ (units minor for 1 unit major currency). For simplicity we consider the same schedule for all legs. We can establish the following relationship:

Proposition 5.1. *When collateral is in a major currency, the following equation holds:*

$$(S^Y_{0,b} + b^{ccs}_{0,b}) \sum_{k=0}^{b} \tau^Y_k P^{D^X}(0, T_i) \frac{\mathcal{X}_0}{\mathcal{F}(t, T_i)} + P^{D^X}(0, T_b) \frac{\mathcal{X}_0}{\mathcal{F}(0, T_i)} = (\mathbf{Float}^X(0, 0, T_b) + P^{D^X}(0, T_b)) \tag{87}$$

Proof. Consider the following portfolio containing a set of transactions and its associated cash flows, without taking the account the effects of collateral, for a swap starting at time $t = 0$, and with all $\tau = 1$ to lighten the notation:

Transaction	Cash Flow Today	Interim Cash Flow	Terminal Cash Flow
Receive fixed on fgn swap	0	$S^Y_{0,b} - L^X_t$	0
X-Crncy Basis Foreign flows	-1	$L^Y_t + b^{ccs}_{0,b}$	+1
X-Crncy Basis Dollar flows	$1/\mathcal{X}_0$	$-L^X_t/\mathcal{X}_0$	$-1/\mathcal{X}_0$
Spot FX Fgn	1	0	0
Spot FX USD	$-1/\mathcal{X}_0$	0	0
Fwd Sale Fgn	0	$-(S^Y_{0,b} + b^{ccs}_{0,b})$	-1
Fwd Buy USD	0	$(S^Y_{0,b} + b^{ccs}_{0,b})/\mathcal{F}(0,t)$	$1/\mathcal{F}(0,T_b)$
Subtotal Float	0	$-L^X_t/\mathcal{X}_0$	$-1/\mathcal{X}_0$
Subtotal Fixed	0	$(S^Y_{0,b} + b^{ccs}_{0,b})/\mathcal{F}(0,T_b)$	$1/\mathcal{F}(0,T_b)$

By assumption, since collateral from all contracts is posted in major currency, the collateral of net cash-flows is also posted in major currency. These net cash-flows resemble those a swap of major currency LIBOR against a schedule of fixed payments. Therefore, by the results we have derived above for **IRS**, the net stated cash flows can be valued by discounting them at the major currency risk-free rate.

The present value of paying LIBOR on a notional $1/\mathcal{X}_0$ and a terminal payment of $1/\mathcal{X}_0$ is obviously $-\frac{1}{\mathcal{X}_0}(\mathbf{Float}^X(t,0,T_b) + P^{D^X}(t,T_b))$. Since there are no net cash flows at time $t = 0$, the total present value of the two subtotal cash-flows must equal 0. Therefore:

$$(S^Y_{0,b}+b^{ccs}_{0,b})\sum_{k=0}^{b}\tau^Y_k P^{D^X}(t,T_k)\frac{1}{\mathcal{F}(0,T_k)}+P^{D^X}(0,T_b)\frac{1}{\mathcal{F}(0,T_b)}-\frac{1}{\mathcal{X}_0}(\mathbf{Float}(0,0,T_b)+P^{D^X}(0,T_b)) = 0 \quad (88)$$

Multiplying all terms by $\mathcal{X}_0$, to express everything in minor currency units, and rearranging, yields the desired result. □

Remark 5.1. *From the analysis we have conducted above for standard* **IRS** *collateralized in some other currency, the fair swap rate $S^Y_{a,b}$ can be considered as independent from the choice of the collateral currency, since the* **LVA** *is not affected by that. We already stressed in the remark 4.2 that it may be possible that the fair* **FRA** *and* **IRS** *rates may be different accordingly to the choice of the currency used to post the collateral. So, if in the market different* **IRS** *are quoted for the different possible currencies for the collateral, we can use these quotes for $S^Y_{0,b}$ in equation (87). Otherwise we can quite safely assume that swap rates for* **IRS** *collateralized in minor currency are the same for any other collateral currency.*

It is straightforward to derive the present value of receiving Libor rates in minor currency from a major currency perspective: since foreign par swaps are fair the value of the fixed cash flows must equal the value of the floating cash flows. The value of the fixed cash flows, from (87), is $S^Y_{0,b}\sum_{k=0}^{b}\tau^Y_k P^{D^X}(t,T_k)\frac{\mathcal{X}_0}{\mathcal{F}(0,T_k)}$, so that:

$$\begin{aligned}\mathbf{Float}^Y(t,0,T_b) = S^Y_{0,b}\sum_{k=0}^{b}\tau^Y_k P^{D^X}(t,T_k)\frac{\mathcal{X}_0}{\mathcal{F}(0,T_k)} = \\ \mathbf{Float}^X(t,0,T_b) + P^{D^X}(0,T_b) - b^{ccs}_{0,b}\sum_{k=0}^{b}\tau^Y_k P^{D^X}(t,T_k)\frac{\mathcal{X}_0}{\mathcal{F}(0,T_k)} \\ - P^{D^X}(0,T_b)\frac{\mathcal{X}_0}{\mathcal{F}(0,T_b)} = \sum_{k=0}^{b}\frac{P^{D^X}(t,T_i)\mathcal{X}_0}{\mathcal{F}(0,T_i)}\tau^Y_k E^{T_k}_{D^X}[L^Y_k(T_{k-1})]\end{aligned} \tag{89}$$

So we are able switch to the major currency T_k-forward measure for each minor currency Libor rate. This allows us to price a **CCS** collateralized in major currency, since we can replace in (86) equation (89), setting $t = T_a = 0$:

$$\begin{aligned}\mathbf{CCS}^X(t, b^{ccs}_{a,b}; 0, ..., T_b) = &\Bigg[\mathcal{X}_0\Bigg(\sum_{k=0}^{b} P^{D^Y}(0,T_k)\mathbf{Float}^Y(0,0,T_b) + b^{ccs}_{0,b} \\ &+ 1 - P^{D^Y}(0,T_b)\Bigg) \\ &- \mathbf{Float}^X(0,0,T_b) + 1 - P^D(t,T_b)\Bigg] \\ &+ \mathbf{LVA}_{\mathbf{CCS}^X_{(0;0;T_b)}} + \mathbf{FVA}_{\mathbf{CCS}^X_{(0;0;T_b)}}\end{aligned} \tag{90}$$

The **LVA** is defined as:

$$\mathbf{LVA}_{\mathbf{CCS}^X_{(0;0;T_b)}} = \sum_{j=1}^{N} P^D(t,t_j)\Bigg[E^{t_j}_{D^X}[\tau^C_j Q^X_j(t)]E^{t_j}_{t_j}\Bigg[\gamma\mathbf{CCS}(t_j;0,T_b)\Bigg] \tag{91}$$

where the notation is the same as above. The **FVA** is defined similarly as:

$$\mathbf{FVA}_{\mathbf{CCS}^X(0;0,T_b)} = \sum_{j=1}^{N} P^{D^X}(0,t_j)\Bigg[E^{t_j}_{D^X}[\tau^C_j U^X_j(t)]E^{t_j}_{D^X}[\gamma\min(\mathbf{CCS}(t_j;0,T_b),0)]\Bigg] \tag{92}$$

Remark 5.2. *The set-up we have outlined above allows a consistent valuation of* **IRS***s in different currencies and of the* **CCS***. From* (87) *one can derive the term structure of the implied FX swap levels. These are then used in* (89) *to bootstrap* **YYY** *Libors seen from the* **XXX** *economy perspective: they guarantee that the* **CCS***s are repriced correctly. So in this approach one does not build a basis adjusted discounting curve to match* **CCS***s' prices, a method usually adopted in practice in many banks. We prefer to build, in our opinion more consistently, adjusted Libor projection curves and leave unchanged the discounting curves. By definition, the* **IRS***s in the two currencies are correctly repriced when using the proper discounting and projection curves.*

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