Research Paper Series

Risk Attribution

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Executive Summary

The well-known Value at Risk (VaR) is the standard market risk quantification for financial institutions. Despite its wide and consolidated usage, evaluating VaR and its decomposition along the desired risk components in a portfolio is still a complex exercise. Commonly, banks do not have automatic risk decomposition frameworks: drill-downs are performed through P&Ls analysis and proxies with sensitivities. Risk controllers and management lack then of a fast and easily readable attribution of VaR (or other risk metrics) to marginal components at both position and risk factor levels. Leveraging on the last two decades of literature on the topic, our approach seeks to provide with a risk decomposition framework which is able to spot (additive) risk and diversification sources within the portfolio(s) at hand that can be beneficial for many goals: risk analysis, uncovering of new (risk-adjusted) business opportunities, improved capital allocation and increased soundness for auditing and supervisory purposes.

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He holds a degree in mathematics, a master degree in mathematical finance and a PHD in statistics. He worked as an executive in both large and regional italian banks. He has also collaborated with some consultant companies in the broad area of risk management, asset management, pricing models, software systems and regulatory compliance. Along with his professional activities, he always develops applied research in the above fields, with about 20 papers published in scientific journals and dozens of speeches in international conferences. He currently gives seminars and lessons in some top ranked italian universities.

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He graduated in Mathematical Engineering at Politecnico di Milano and now he is a quantitative analyst in Iason. He has been working in a project with one of the major Italian banks, providing first support to the risk monitoring team and now developing methodologies for the decomposition of risk measures.

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Business Analyst
He graduated in Quantitative Finance at Bocconi University. As Business Analyst, he is currently working on the development of risk analysis and attribution methods and the IT implementation of a VaR decomposition engine for one of the major Italian banks.

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The VaR is only one of the possible realizations of the unknown underlying distribution of the portfolio Profits and Losses (P&Ls), and its calculation has been addressed with different approaches, from parametric to fully non-parametric, as in simulations settings. Its decomposition to the risk component's contributions is then even a more subtle task. Many are the works that in the years tried to shed light in how VaR and its factorization can be computed efficiently and reliably. References as, for example, [6], [7] and [1] show how to compute robust quantiles. Other works, like [4], [5] and [8], assessed instead the task of decomposing risk metrics such as VaR into the single portfolio's risk components. Our approach, which builds upon some of these works, tries to provide a framework to:

- Properly decompose the portfolio VaR into easily readable position and risk factor's components;
- Perform what-if analysis without recall to full simulations, but rather exploiting marginal contributions to evaluate market stresses (e.g., interventions on position and risk factor's values) and trading activity (e.g., addition or removal of positions in the portfolio).

Risk Attribution: Key Concepts and Challenges

In our work we refer to *Risk Attribution* as the decomposition of VaR (at a given confidence level α and holding period) into the risk contributes at either position or risk factor level such that VaR can be valued as the algebraic sum of marginal components. This task is quite subtle for many reasons:

- VaR is not a sub-additive measure, hence we cannot calculate the VaR of each desired component and then add up to the diversified global measure;
- The true P&L distribution is unknown and we can only leverage on parametric or empirical proxies;
- Portfolios' P&Ls are usually fat-tailed, adding complexity in the estimation of the risks scenarios;
- Splitting VaR into positions has to allow to correctly identify the risky and the hedging positions, and decomposing on the risk factors adds up complexity introducing many non-linear and subtle second order effects which are tricky to handle and estimate;
- Even if the probabilistic model was perfectly known, the popular risk decomposition measures (such as the Component VaR) would show a higher statistic instability compared to the respective portfolio VaR.

Regarding the last bullet point, we show here a simple example to highlight the statistical problems in the estimation of the VaR decomposition. We assume the followings:

- 3 assets $(p_i)_{i=1,2,3}$ with returns that are i.i.d. normally distributed with known volatility σ;
- Weights $(w_i)_{i=1,2,3}$ equal to $1/3$ for each of the 3 assets.

Risk Estimation			
Error Par	Rel. StDev Err(Par)	Error HS	Rel. StDev Err(HS)
-0,44%	4,34%	-3,52%	6,95%

Risk Attribution		
	Mean (Error)	Rel. StDev (Error)
$CVaR_1$	-0,36%	9,13%
$CVaR_2$	0,17%	8,96%
$CVaR_3$	0,19%	9,15%

TABLE 1: Simulation results

Given the assumptions above, the portfolio VaR over a time horizon T is simply given by:

$$VaR\,(\alpha, h) = \sigma_{PTF} \cdot z_{\alpha} \cdot \sqrt{T}.$$

By symmetry, the correct Component VaR (CVaR) for each asset $(p_i)_{i=1,2,3}$ is given by:

$$CVaR_i = \frac{1}{3} VaR_{PTF}.$$

In the Tables 1, we show some results where the VaR and $CVaR_i$ are estimated in different ways (in a Montecarlo simulation framework):

- **Parametric approach**: VaR estimation by volatility, assuming that the distribution is known;
- **Historical simulation**: VaR estimation via empirical quantile, where no assumptions on the distribution is required;
- **CVaR by Garman**: VaR decomposition via linear regression [4].

Here we used $N = 1000$ for the number of simulations, to identify the statistic uncertainty of the estimation. We can see some results in the Tables 1.

The true uncertainty of the model is given by the standard deviation of the error in its relative version, namely divided by the correct risk measure (unknown to the modeler). The last row shows that the attribution error is greater than the error in the global VaR estimation (standard deviation in the first row of the Table 1).

Benefits and Applications

Risk attribution frameworks, more common in the asset management industry, can be a great boost for both risk analysis and day-to-day business decisions for banks, and help to stick with reporting requirements of both BCBS 239 (Risk Data Aggregation and Reporting Principles) and CRR. Summarizing, benefits and applications of such a framework can be seen in:

- **Risk analysis**, simplifying portfolio inspections and sensibly increasing the efficiency of risk vs hedge sources' identification;
- **Business opportunities** discovery, helping management to boost risk-adjusted portfolios' performance;
- **Capital allocation** optimization, with risk managers and traders who can assess how to balance different trade opportunities with the related cost;

- **Auditing and supervisory**, with the framework useful in the discussions with internal and external surveillance departments and authorities, helping to assess portfolio composition and relative risks and costs.

1. Methodology

Our approach builds, as anticipated, on a wide literature (see also [9] and [10] to cite other works) and tries to extend the results in at least two directions:

- Modeling the P&L distribution w.r.t. the component P&Ls' distribution, in order to have a model more scalable for the valuation of different percentiles at once;
- Easily readable decomposition of the VaR in additive risk factors' components.

1.1 The General Setting

To describe the approach to the *Value at Risk (VaR)* decomposition in the so-called *Risk Attribution* framework, let's start assuming to hold a portfolio $\mathcal{P} = \{p_1, ..., p_N\}$ composed by a set of positions p_n such that the value of $\mathcal{P}$ today, its market value, is simply given by:

$$P_0 = \sum_{n=1}^{N} p_n(t_0) = \sum_{n=1}^{N} p_n^0,$$

where, to simplify and with a small abuse of notation, p_n^0 is the net present value of the n-th position today[1], t_0. Assume further that $\mathcal{P}$ insists on a set of risk factors $\mathcal{R} = \{r_1, ..., r_K\}$ such that each position p_n depends on a subset $\mathcal{R}_n = \{r_{n_1}, ..., r_{n_{k_n}}\} \subseteq \mathcal{R}$ of these drivers:

$$p_n^0 = f(r_{n_1}, ..., r_{n_{k_n}})$$

for some pricing function[2] $f(\cdot)$ and integer k_n, where r_{n_j} is the j-th risk factor for the n-th positions in $\mathcal{P}$. Hence, assuming for the sake of simplicity a fixed portfolio composition, a variation of the portfolio value in a time span τ can be expressed as:

$$\begin{aligned}\Delta P^{\tau} &= \sum_{n=1}^{N} p_n^{t+\tau} - p_n^t \\ &= \sum_{n=1}^{N} f(r_{n_1}^{t+\tau}, ..., r_{n_{k_n}}^{t+\tau}) - f(r_{n_1}^{t}, ..., r_{n_{k_n}}^{t}),\end{aligned}$$

where $r_{n_j}^t$ refers to the value of the j-th risk factor of the n-th positions at time t. In the usual bank's market risk valuation process, the VaR of $\mathcal{P}$ with confidence level α is obtained either by Monte Carlo or through historical simulation with S scenarios, where in the latter S usually is 250 or 500 (one or two years of history respectively), and with a time horizon of one day. In our notation:

$$VaR = \Delta P^{\hat{s}},$$

where $\hat{s}$ is the scenario index (with $\hat{s} \in \{1, ..., S\}$) such that $\mathbb{P}[\Delta P^s \leq \Delta P^{\hat{s}}] = 1 - \alpha$ for every other scenario s.

1.2 Risk Attribution: Definition and Methodology

With *Risk Attribution* we refer to an additive decomposition of the VaR metric at a given level (e.g., trading portfolio level) into its basic components c_i (trades, p, or risk factors, r):

$$VaR = \Delta P^{\hat{s}} = \sum_i \phi(c_i),$$

[1] We will refer later with p_n^t to the market value of the n-th position at time t

[2] For simple cash positions as equities or funds, trivially $f(\cdot) = Q \cdot M$, where Q is the position's quantity and M is the price (NAV respectively) on the market

where any term of the summation can uniquely be attributed to each of the individual components c_i through an appropriate mapping function $\phi(\cdot)$.
The approach here proposed has roots in the works of [4] and [5] where VaR is decomposed at position level exploiting the:

- Very definition of conditional expectation;
- Fact that the portfolio (euro) return is as a linear combination of the (euro) returns on the individual components.

1.3 MVaR, CVaR and IVaR

Assuming that z^s is the return (i.e., shock under scenario s) of the portfolio and z_n^s the same but at position level (i.e., $p_n^s = p_n^0 z_n^s$), we get:

$$\begin{aligned} VaR &= \Delta P^{\hat{s}} = \mathbb{E}[\Delta P^{\hat{s}} | z^{\hat{s}}] \\ &= \sum_{n=1}^{N} \mathbb{E}[p_n^0 z_n^{\hat{s}} | z^{\hat{s}}] = \sum_{n=1}^{N} p_n^0 \mathbb{E}[z_n^{\hat{s}} | z^{\hat{s}}], \end{aligned} \tag{1}$$

with this representation, one can note as:

- $\mathbb{E}[z_n^{\hat{s}} | z^{\hat{s}}]$ represents the *Marginal VaR* for the n-th position, $MVaR_n$, hence its contribute to the total VaR if it were a unitary position;
- $p_n^0 \mathbb{E}[z_n^{\hat{s}} | z^{\hat{s}}]$ is, instead, the *Component VaR* for the n-th position, $CVaR_n$, hence the true VaR contribute given the size of the position.

Building once more on the works [4] and [5] the *Incremental VaR*, $IVaR_n$, due to a new position p_{N+1} entering the portfolio $\mathcal{P}$ can be evaluated by first order Taylor series expansion:

$$VaR' \approx VaR + p_{N+1}^0 MVaR_{N+1},$$

where $MVaR_{N+1}$ is calculated w.r.t. the original portfolio $\mathcal{P}$, not to the augmented $\mathcal{P}' = \mathcal{P} \cup \{p_{N+1}\}$; this way we get:

$$IVaR_{N+1} = p_{N+1}^0 MVaR_{N+1}. \tag{2}$$

This approach can be extended to the case where several assets are added to (or removed from) the portfolio at the same time:

- If one or more positions are added to the original portfolio $\mathcal{P}$, the marginal VaRs of these must be evaluated upfront w.r.t. $VaR = VaR(\mathcal{P})$ by means of the contribute of dummy P&Ls added to the overall portfolio P&L, but to whom the original positions' market value is set to 0 (i.e., $p_j^0 = 0$ for all positions p_j to be added to $\mathcal{P}$); this way the original VaR can be still reconstructed from the only trades in $\mathcal{P}$: $VaR = \sum_{n=1}^{N} CVaR_n + \sum_{m=1}^{M} 0 \cdot MVaR_m$.
- If one or more positions p_j were instead to be removed from $\mathcal{P}$, one must set $p_j^0 = 0$ for all of them.

From now on we will focus on CVaR as the most useful definition of Risk Attribution:

$$VaR = \Delta P^{\hat{s}} = \sum_{n=1}^{N} CVaR_n.$$

1.4 Decomposition at Risk Position Level

As outlined in the previous section, the risk attribution model must evaluate the conditional expectation $\mathbb{E}[z_n^{\hat{s}} | z^{\hat{s}}]$ (cfr. Eq. 1) and to do so we must link the portfolio return $z^{\hat{s}}$ with each single position, $z_n^{\hat{s}}$. Due to the additive linear relation between the Profits and Losses (P&Ls) of the positions and the portfolio's one, our approach consists in estimating the conditional expectation

through linear regression. This allows to capture the interaction between positions' P&L and the ones of the portfolio at a much broader scale than the single percentile of interest (e.g., the VaR), ensuring a more versatile use of the model as discussed shortly after.
Now, let:

- $x = (x_s)_{s=1,...,S}$ be the array of P&Ls at portfolio level, where $x_s = P^s - P^0 = P^0 z^s$ is the P&L for the entire portfolio at scenario s;
- For each position p_n, $y_n = (y_{n,s})_{s=1,...,S}$ be the array of P&Ls at position level, where $y_{n,s} = p_n^s - p_n^0 = p_n^0 z_n^s$ is the P&L for the position p_n at scenario s.

We then calibrate a set of N regressive models $f_n(\cdot)$ of the form:

$$z_n = f_n(z) = \beta_n z + \epsilon_n, \tag{3}$$

where $z = (z_s)_{s=1,...,S}$ and $z_n = (z_{n,s}.)_{s=1,...,S}$ are, respectively, the set of portfolio and positions' returns under scenario. Note that we truncated the canonical intercept from Eq. 3, namely α_n, since in a daily[3] horizon setting the drift effect is negligible. Once we have calibrated the set of models f_n, the VaR of $\mathcal{P}$, $\Delta P^{\hat{s}}$, can be decomposed as follows:

$$VaR = \Delta P^{\hat{s}} = \texttt{Dec}(p_1, ..., p_n; \hat{s}) = \sum_{n=1}^{N} p_n^0 f_n(z_{\hat{s}}), \tag{4}$$

where:

- $f_n(z_{\hat{s}})$ is the Marginal VaR, $MVaR_n$, for the n-th position;
- $p_n^0 f_n(z_{\hat{s}})$ is the Component VaR, $CVaR_n$, for the n-th position.

The Incremental VaRs, $(IVaR_j)_{j \in J}$, can be calculated instead as in Eq. 2.

1.5 Decomposition at Risk Factor level

In this paper, with risk factor level we refer to any possible cluster of elemental risk factors; indeed, it is not only possible to work at an atomic level, but also at any higher order of aggregation (e.g., Asset Class).
The decomposition of the VaR on the risk factors $\mathcal{R} = \{r_1, ..., r_K\}$ leverages on a approach similar to the one just explained , but must account for further complexities:

- The relation between portfolio P&Ls and risk factor scenarios is not strictly linear, especially for a complex trading book like major banks have;
- Risk factors are not only scalars as the P&Ls under scenarios (i.e., each P&L value under a specific scenario is a scalar value), but more often can be arrays (e.g., interest rate curves) and matrices (e.g., equity volatilities) or even cubes (e.g., swaption volatilities).

Our proposal, to balance the trade off between the complexity of the model, its readability, and the effort potentially required to maintain the data model beneath, builds upon the canonical P&L explanation through the portfolio sensitivities and three major steps of data processing.

1.5.1 P&L Explanation

Given the set $\mathcal{R}$ of portfolio's risk factors, let:

- r_k^s be the scenario s for the risk factor r_k;
- δ_k be the market sensitivities related to r_k.

[3] The holding period we are interested in for a canonical bank VaR valuation framework.

Similarly to what we set for both portfolio and positions' market values, we can write the risk factor scenario as a function of the present value of the risk factor and its shock under scenario:

$$r_k^s = \varphi(r_k^0, b_k^s),$$

where:

- r_k^0 is the present value of the risk factor r_k on the market;
- b_k^s is the shock (i.e., bump/ shift) to be applied on the k-th risk factor on the scenario s;
- $\varphi(\cdot)$ is the function which defines the scenario value, that can be either multiplicative (e.g., $\varphi(r_k^0, b_k^s) = r_k^0 b_k^s$ as for the exchange rates) or additive (e.g., $\varphi(r_k^0, b_k^s) = r_k^0 + b_k^s$ as for the interest rate curves).

The portfolio P&Ls under scenarios can then be recalculated as a proxy with risk factor scenarios and sensitivities through a Taylor expansion:

$$\begin{aligned} \Delta P &= (\Delta P_s)_{s=1,\dots,S} \\ &\approx \sum_{k=1}^{K} b_k \delta_k = \left(\sum_{k=1}^{K} b_k^s \delta_k \right)_{s=1,\dots,S}, \end{aligned} \tag{5}$$

where, in the first sum, b_k must be intended as the array of scenarios for the k-th risk factor. Note that, in the notation of Eq. 5, we assumed for the sake of simplicity that b_k^s and δ_k are both scalars: the reason will be clear in a while and, with just a little more algebra, the P&L proxy above remains valid in the more general case.

1.5.2 The Risk Factors Preprocessing

As mentioned before, our model handles the complexities due to the different geometry of risk factors and to their nonlinear effects on the portfolio P&Ls through three main steps:

- Restricting the model to the tail of the P&Ls' distribution, rather than regressing the entire P&Ls as in 3; for example, regressing only on the last 10% of scenarios on the risky tail;
- Building meta (and scalar) risk factors $\widetilde{\mathcal{R}} = \{\widetilde{r}_1, \dots, \widetilde{r}_K\}$; for example, instead of regressing the entire EURIBOR 6M curve, we opted to build a synthetic driver that embeds the contributes of all the curve pillars;
- Possibly restricting the risk factor covariates only to a selection of the main drivers in the portfolio under analysis; e.g., selection based on the market sensitivities.

In particular, the second point recalls the simplification expressed in Eq. 5 where we supposed scalars risk factors build as explained below. Assume, for example, that r_k is an array (e.g., a curve) which implies δ_k, the array of the market sensitivities related to r_k, be an array too. We propose to build the meta risk factor $\widetilde{r}_k$ as the sum of all the element-wise multiplications between the components of r_k^s and δ_k; in our example (i.e., the curve) we obtain:

$$\widetilde{r}_k^s = \sum_j r_{k,j}^s \delta_{k,j}.$$

The same idea holds for matrices and cubes, just with more calculations. With this approach we aim at:

- Incorporating the dependencies (in terms of shocks' direction and magnitude) of the portfolio to the risk factors directly into the meta risk factors;
- Reducing the complexity of the decomposition of the VaR, recalling to simpler covariates in the regression problem;

- Induce the covariates of our regressive approach (see next paragraph) to be robust in terms of magnitude and direction with the expected P&Ls (and hence, Component VaRs); given a threshold ϵ, the following should hold:

$$\mathtt{dist}(CVaR_k - \delta_k b_k^{\hat{s}}) \leq \epsilon,$$

  for an appropriate (e.g., Euclidean) distance measure and

$$\mathtt{sign}(CVaR_k \cdot \delta_k b_k^{\hat{s}}) = +1.$$

1.5.3 The Model

Let assume now that risk factors r_k are already meta (and scalar) risk factors as explained above. Then, the risk attribution model leverages on the P&L proxy expressed in Eq. 5 and recalls once more the approach in Eq. 3 that now reads as follows:

$$b_k \cdot \delta_k = f_k(x) = \beta_k x + \epsilon_k,$$

where $b_k \cdot \delta_k$, as previously defined, is the array of (proxy) components of portfolio's P&Ls and we regressed this time directly on the portfolio's P&Ls. Given this, the model decomposes the VaR for the portfolio $\mathcal{P}$ as:

$$VaR = \Delta P^{\hat{s}} = \mathtt{Dec}(r, ..., r_k; \hat{s}) = \sum_{k=1}^{K} A \cdot \beta_k x_{\hat{s}},$$

where A is an adjustment factor defined as:

$$A = \frac{\Delta P^{\hat{s}}}{\sum_k \beta_k x_{\hat{s}}}.$$

Note that A is needed to amend for the fact that, differently from the decomposition for position, the P&L proxies by risk factor does not perfectly sums to the portfolio's P&L.

2. CVaR Estimation Methods

To understand at a deeper level the advantages of using the methodology described in the previous sections, we will see now a comparison between the results of three different methods for the CVaR estimation:

- **CVaR estimation by *local P&L*:** the CVaR of the single position is given by the corresponding P&L in the VaR scenario of the portfolio;
- **CVaR estimation by Garman:** as anticipated in section [2.5], we calibrate the regression model for each position to obtain the CVaR as in Eq. 4;
- **CVaR estimation with Harrell-Davis weights:** the CVaR of each position is given by a weighted average of its P&Ls; the weights are given by the Harrell-Davis formula (as we will show later in section [3.1]), where the scenario with higher weight will be the VaR scenario of the portfolio.

2.1 Estimation with Harrell-Davis Weights

Before introducing the specific Harrell-Davis weights that we will use in this case, we will depict the concept of L-estimators (references [1], [2], [6] and [7]).
We will assume that the random variable L represents the loss of the portfolio, with C.D.F F, namely $\mathbb{P}(L \leq x) = F(x)$; the portfolio's VaR of order α ($0 < \alpha < 1$) will be given by $VaR_\alpha = F^{-1}(1-\alpha)$. Given the set of scenarios $(s)_{s=1,...,S}$, the set of losses $(L_s)_{s=1,...,S}$ will provide an empirical approximation of the true loss distribution denoted by F; from the empirical distribution we will obtain the estimated value for VaR_α that we will denote with $\mathcal{Q}_\alpha$.

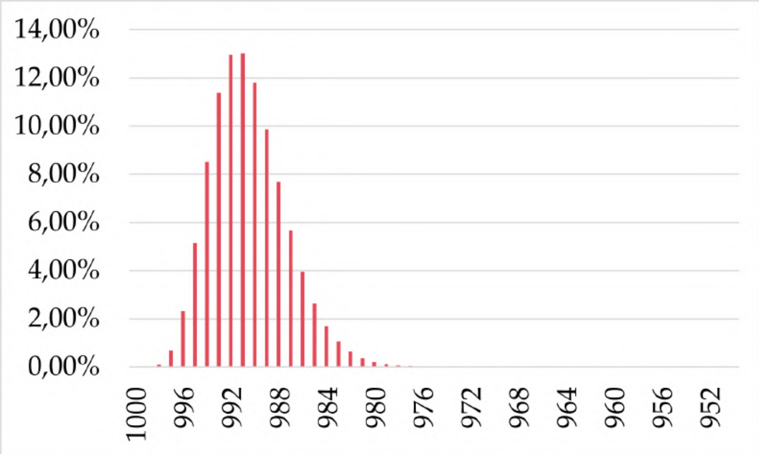

FIGURE 1: *Weights for Harrell-Davis estimator with* $\alpha = 99\%$ *and* $S = 1000$.

For example, one of the most common non-parametric quantile estimator is the sample quantile, also known as the *upper empirical cumulative distribution function value* (UECV), namely the k-th largest loss. The main problem with UECV is that it relies on a single order statistic, hence it can exhibit high variability, reducing the efficiency.
A more generalized form for a L-estimator is the following (weighted average of the statistics):

$$\mathcal{Q}_\alpha = \sum_{s=1}^{S} w_{\alpha,S,s} \cdot L_s$$

$$with \sum_{s=1}^{S} w_{\alpha,S,s} = 1.$$

The UECV introduced above would lead to the following weights:

$$w_{\alpha,S,s} = \begin{cases} 1 & if \quad s = \lfloor S\alpha \rfloor + 1 \\ 0 & otherwise. \end{cases}$$

To improve the efficiency, L-estimators generally calculate $\mathcal{Q}_\alpha$ as a weighted average of multiple order statistics. For example, the Harrell-Davis estimator is based on the fact that the expected value of order statistic $(S+1)\alpha$, namely $\mathbb{E}[L_{(S+1)\alpha}]$, converges to $F^{-1}(\alpha)$. The weights in this case are given by:

$$\begin{aligned} w_{\alpha,S,s} &= I_{s/S}\big((S+1)\alpha, (S+1)(1-\alpha)\big) \\ &\quad - I_{(s-1)/S}\big((S+1)\alpha, (S+1)(1-\alpha)\big) \\ &= \frac{1}{B\big((S+1)\alpha, (S+1)(1-\alpha)\big)} \\ &\quad \int_{(s-1)/S}^{s/S} u^{(S+1)\alpha-1}(1-u)^{(S+1)(1-\alpha)-1}\, du, \end{aligned}$$

where:

- $I_x(a,b)$ is the Regularized Incomplete Beta Function, defined as:

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)};$$

- $B_x(a,b)$ is the Incomplete Beta Function, defined as:

$$B_x(a,b) = \int_0^x u^{a-1}(1-u)^{b-1}\, du;$$

	Asset1	Asset2	Asset3	Asset4	Asset5	Asset6	Asset7	Asset8	Asset9	Asset10
CVaR (analytical)	-1.388%	-1.388%	-1.388%	-1.388%	-1.388%	-1.388%	-1.388%	-1.388%	-1.388%	-1.388%
CVaR (local)	-0.116%	-1.392%	-2.633%	-1.670%	-1.755%	-2.634%	-0.658%	-2.616%	-0.876%	0.472%
CVaR (Garman)	-1.261%	-1.522%	-1.379%	-1.381%	-1.424%	-1.362%	-1.316%	-1.397%	-1.466%	-1.369%
CVaR (Harrell-Davis)	-0.942%	-1.799%	-1.511%	-1.479%	-1.684%	-1.758%	-0.910%	-1.435%	-1.402%	-0.957%

TABLE 2: Simulation results

- $B(a, b)$ is the Beta Function, defined as:

$$B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\, du = B_1(a,b).$$

In the Figure 1, we can see the distribution of the Harrell-Davis weights given $\alpha = 99\%$ and $S = 1000$, namely the same values we used in the example that we will see in the section 'Results Comparison'.

It is worth mentioning that other L-estimators can be used: among them one with very promising performances is the Epanechnikov's, for more details see [11]. We will investigate its application in further detailed studies.

2.2 Results Comparison

In this section we will show a practical example to compare the three methodologies introduced in the previous section.
In the exercise we assume the followings:

- 10 assets $(p_i)_{i=1,\ldots,10}$ with returns that are i.i.d. normally distributed with volatility $\sigma_{daily} = 1\%$;
- Correlation $(\rho_{i,j})_{i,j=1,\ldots,10}$ equal to 0.5 for each pair of assets;
- Weights $(w_i)_{i=1,\ldots,10}$ equal to 10% for each asset;
- 1000 scenarios generated via Montecarlo simulation.

In this simple example, by symmetry, the $CVaR$ of each asset should be equal to 10% of the VaR value. In the specific simulation shown here we obtained a portfolio VaR of -13.88% and the following results for the $CVaR$ of the 10 assets in the three methodologies.
It is obvious that given the Montecarlo framework, each run will lead to different results, but we will always see an higher instability with the local approach compared to the other two.
In the first row we can see the "true" $CVaR$ which is equal to $1/10 \cdot VaR$, namely -1.388% for each asset; this will be our reference for the comparison of the other three rows. The main results we can deduce are the followings:

- The local $CVaR$ leads to a huge instability, we even have a positive $CVaR$ in the last asset;
- The $CVaR$ obtained with Harrell-Davis weights is way more stable compared to the local one. This was an expected result as explained in section [3.1];
- Lastly we can see how the $CVaR$ by Garman introduced in section [2.5] is the one that gets closer to the analytical $CVaR$ in the first row.

3. Conclusion

Even though in recent years, not many papers have been published displaying VaR decomposition methodologies, we decided to explore these advanced inferential statistical methods, such as [4], [6] and [11], mainly for their trade-off between accuracy and computational time.
Normally, the Component VaR (CVaR) of each position is estimated via full-valuation Incremental VaR (IVaR), i.e. the difference between the original portfolio VaR and the VaR of the portfolio without the given position. The problem with this method is not only the computational time needed to

obtain the CVaR of each position, but also the fact that to obtain the CVaR of a cluster of positions one needs to compute a VaR for each of the combination one wants to analyze. This implies even thousands of different sub-VaR calculations, which are not feasible for a daily risk management process. Instead of using a brute-force approach, the implementation of the advanced inferential statistical methods mentioned above allows to save a lot of computational effort, by avoiding a lot of valuations (pricing tasks), and still achieves very accurate risk decomposition figures. On the other hand, using for example the Garman method, thanks to the linearity of the decomposition you will be able to compute the CVaR of each cluster by simply summing the atomic CVaR.

Finally, we presented and discussed a possible solution to apply the above methodologies to obtain a VaR decomposition by risk factor. It is worth to remind that in the common practice risk management departments just get a VaR decomposition by position (or portfolio). Here the main problems are given by the non-linear relation between portfolio P&Ls and risk factor scenarios and by the fact that risk factors are not only scalars. In order to overcome these problems we used the sensitivities for a first order Taylor expansion of the P&Ls, checking very carefully the accuracy of the technique against a full-valuation approach.

In the usual trade-off between accuracy and computational effort, the results seem to be very encouraging.

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