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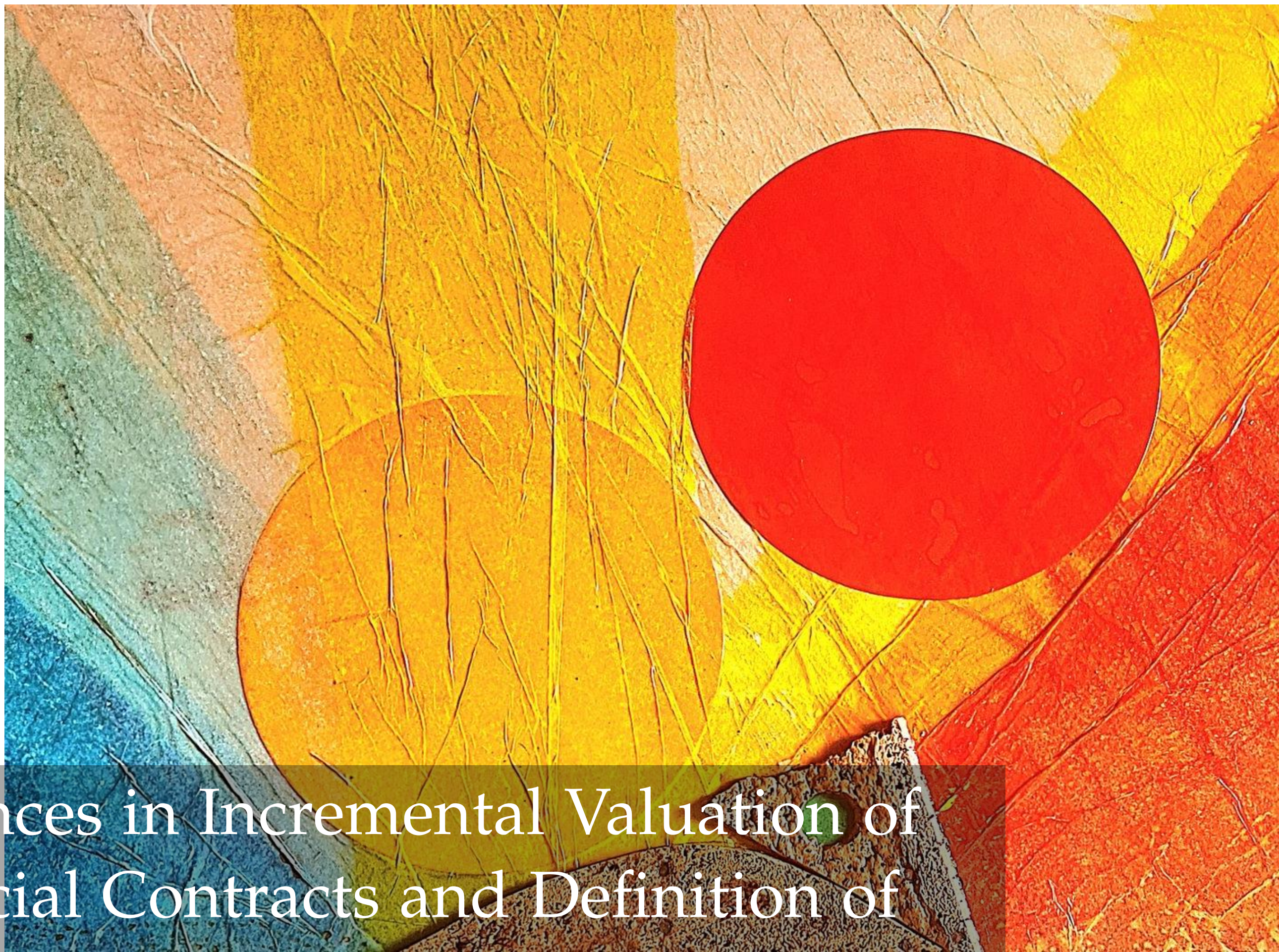

Advances in Incremental Valuation of Financial Contracts and Definition of the Economic Meaning of the Capital Value Adjustment (**KVA**)

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Executive Summary

We extend the analysis we sketched in Castagna [5] and we provide an application of the framework we introduced to incrementally evaluate financial contracts within a financial institution's balance sheet.

This work draws on the presentation at the GlobalDerivatives Conference of May 2015, in Amsterdam (available here). This is a preliminary version. Comments are welcome.

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Antonio Castagna is currently managing partner and founder of the consulting company Iason. He previously was in Banca IMI, Milan, from the 1999 to 2006: there, he first worked as a market maker of cap/floor's and swaptions; then he set up the FX options desk and ran the book of plain vanilla and exotic options on the major currencies, being also responsible for the entire FX volatility trading. He started his carrier in the investment banking in the 1997 in IMI Bank, in Luxemborug, as a financial analyst in the Risk Control Department. He graduated in Finance at LUISS University in Rome in 1995, with a thesis on American options and the numerical procedures for their valuation. He wrote papers on different topics, including credit risk, derivative pricing, collateral management, managing of exotic options risks and volatility smiles. He is also author of the books "FX options and smile risk" and "Measuring and Managing Liquidity Risk", both published by Wiley.

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Advances in Incremental Valuation of Financial Contracts and Definition of the Economic Meaning of the Capital Value Adjustment (KVA)

Antonio Castagna

In Castagna [5] we sketched a framework to evaluate a contract inserted within the balance sheet of a financial institution. The main result of that work is the importance to assess the impact that the contract's insertion in the bank's books causes in terms of changes of the value of the bank. This is tantamount to saying that a contract has a value to the bank that equals its incremental (or marginal) contribution to the total net value of the bank.

The first consequence of this approach is that the (incremental) value to the bank is a subjective quantity that does not need to be that same as the price quoted and dealt in the market. The difference between price and value of a contract is a concept that we stressed in Castagna [4] and [3]: For an evaluator that is a hedger/replicator, the price of a (derivative) contract is just the payment terms that both parties agree upon when closing the deal; the value of the same (derivative) contract is the present value of the costs paid to replicate the intermediate and final pay-off until the expiry, which in turn is the incremental change in the total hedger/replicator's net value.

The second consequence is that the valuation should be correctly operated by considering the existing balance sheet structure. The main result from the analysis in Castagna [4] is that, if the contract is sufficiently small so that it does not alter (for practical purposes) the probability of default of the evaluator (bank), then the approximated value can be fairly considered as the equivalent to the algebraic sum of the value of i) an otherwise identical risk-free contract (the "pure" value"), ii) the Credit Value Adjustment **CVA** and iii) the Funding Value Adjustment **FVA** referring to the same contract. If the contract has a large notional so that it changes the evaluator's default probability, then the value has to be determined in a more precise fashion and it is given by the algebraic sum of i) an otherwise identical risk-free contract, ii) the Credit Value Adjustment **CVA**, iii) the Funding Value Adjustment **FVA** and iv) the Limited Liability Value Adjustment (**LLVA**). The latter quantity is somehow similar to the more common Debit Value Adjustment (**DVA**), in that it affects the value of the contract in the opposite direction than the **CVA**; nonetheless it cannot be considered as equivalent to the **DVA** for many reasons that we thoroughly discuss in Castagna [4].

From these two considerations, we can easily draw a third one: only the value of a contract can be *incremental*. The concept of *incremental* price is meaningless, because the price cannot include all the incremental valuations referring to the parties involved in the transaction, or two generic parties that would trade in that contract if the price is only a quote not yet dealt. Clearly we are not saying here that the two parties do not consider their own incremental value when bargaining before closing the deal at the agreed price: on the contrary, they will try to push the price as near as possible to the (likely diverging) values they assign to the contract. The effectiveness of this effort depends on the relative bargaining strength existing between the two parties.

After having recalled and stressed the importance of the incremental valuation, it is worth dwelling a little more on this concept to better clarify what the following analysis will aim at. Basically the incremental valuation relies on the recognition that, for a given evaluator, a contract

A cannot be taken in an abstract fashion without considering all the links and interrelations it has with all the rest that is not A, within the reference system.[1]

The problem now shifts to the definition of the boundaries of the reference system, with respect to which the bank evaluates the incremental impact of the new contract it trades. As mentioned above, theoretically speaking, the system should be the economic wealth of the evaluator (*i.e.*: the bank), which is to say: the economic value of the bank to its shareholders. In fact, shareholders are the last claimants on the residual value of the assets, so that the value to them clashes with the ultimate value of the bank, after having considered the payment of all the other stakeholders that have a higher grade in the claimants' order. The only way to compute this value is to jointly evaluate all the assets and liabilities (investments, securities, contracts, etc.), taking into account the limited liability that is granted to the shareholders, to come up with the net wealth of the bank.

We are aware that such task is very difficult and cumbersome, as well as the task to calculate the incremental impact on the value of a new contract dealt by the bank. So, in practice one may limit the analysis to a specific business unit, or area (*e.g.*: the dealing room): this will surely make the calculation more feasible, if heavy yet, but under a theoretical point of view, this limitation of the analysis should be handled with care, lest it produce senseless results.

We would like to stress the fact that if the shareholders directly, or (as it is usually the case) the bank's management indirectly, miximise the bank's value, they are also acting in the best interest of all other senior claimants to the assets' value (*e.g.*: bond holders, depositors, etc.). So, bank's value maximisation begets no conflict of interests between shareholders and other claimants.

Finally, we think we also provide in what follows a perspective to conciliate the apparently theoretical unsound market practices to evaluate derivative contracts, and the nowadays standard results of the modern financial theory, namely the Modigliani-Miller (MM) theorem (see Modigliani and Miller, [9]). Although we think that we already proved the substantial consistency of the current practices in Castagna [4], we will offer below some more thoughts on this matter.

We would like also to mention some recent research in the incremental valuation area: Andersen *et al.* [1] analysed the incremental impact of a contract within a bank's balance sheet, in a framework which fundamentally is the same as in Castagna [4], apart from the markedly more formal mathematical setting. They acknowledge this in mentioning our work, but they claim they reach an opposite result with respect to the number and types of adjustments to consider in the evaluation of a contract. We beg to differ: in reality when they consider the incremental indifference price with respect to the shareholders' value of the bank, they replicate exactly the same result as in Castagna [4]. The differences arise when they consider the impact on the value of the assets of the bank (or, of the whole firm, as it is usually referred to in the literature), disregarding how this incremental impact is distributed amongst all the stakeholders (claimants) on the liability side of the balance sheet.

The last approach is, in our opinion, not correct since it may happen that an indifference contract price, which leaves unaltered the total firm value, entails a gain of one of the stakeholder (*e.g.*: bondholders) and a loss of one or more of the others (*e.g.*: the shareholders). The same may happen even if the contract price increases the whole firm value, but it produces a transfer of value from one stakeholder to another. This is explicitly admitted by Andersen *et al.* [1], when the describe the transfer of value from shareholders to bondholders, but this does not seem to cause any problem to them, in contrast with us; they also appeal to the Modigliani-Miller theorem to justify the results, and we will dwell on that later on.

In the end, we think that when limiting the comparison to the only case that is economically and financially acceptable by all the stakeholders (*i.e.*: the impact of a contract on the value to shareholders), Andersen *et al.* [1] confirm the results in Castagna [4].

[1] The term *abstract* should then be meant in its etymological sense of *abs-* (Latin for *from*) *-tractum* (p.p. of *trahere*, Latin for *to take*), hence: taken away (implied: from the system it belongs to). Although we do not often rely on Hegelian philosophy, its definition of the concept "*abstract*" is indeed one of the few instances where it may offer a useful logical tool (see Hegel, *Encyclopaedia of the Philosophical Sciences*, §§119-120, 3rd ed., 1830).

1. A Continuous-Time Setting for Incremental Valuation of Financial Contracts

We start by generalising the discrete-time setting sketched in Castagna [4], setting up the stage for continuous-time framework for the incremental valuation of financial contracts. Moreover, we need also to work in a general equilibrium framework, since it is not generally possible to determine a single equivalent probability measure based on a no-arbitrage argument that relies on a dynamic replication.

Amongst the many works that in the past have analysed general equilibrium settings, in the following we will refer to that one outlined in Cox, Ingersoll and Ross (CIR, [6]), although we relax some of its assumptions. Namely, we will work in an economy where a single good is produced by means of N production technologies whose transformation process is governed by a system of stochastic processes. Each technology is affected by K state variables Y_k (with $k = \{1, ..., K\}$), whose evolution too is governed by a system of stochastic variables:

$$dY_k(t) = \mu_t^k dt + \sigma_t^k dW_t^k.$$

There is a single interest rate r at which a fixed number of economic agents may borrow or lend but, differently from CIR [6], we allow for the default of the borrower, which means that the terminal pay-off of a loan includes the expected losses due to the borrowers' default. Finally, another assumption of CIR [6] we relax is that all economic agents have an identical utility function *à la* Neumann-Morgenstern: we consider a specific utility function for each economic agent and a general utility function that can be seen as a sort of average of the single agents' utility functions. The possibility to have different utility functions allows the presence of different risk-premia over the risk free rate r within the expected yield of the contingent claims possibly traded in the economy, whose pay-offs may depend on the K state variable and the wealth of a single agent, or of the entire economy aggregate wealth. The relaxing of the assumptions of the CIR [6] setting are necessary to distinguish between the fair (objective) price of a contract (contingent claim) and its (subjective) value.

Let us consider the value of the bank to shareholders[2] at time t, $\mathbf{VB}(t)$: assuming that in the banks' balance sheet there are I assets $A_i(t)$, cash $B(t)$ and J liabilities (debt) $L_j(t)$, the evolution of the value of the bank can be written as:

$$d\mathbf{VB}(t) = \sum_{i=1}^{I} dA_i(t) + dB(t) - \sum_{j=1}^{J} dL_j(t). \tag{1}$$

The dynamics of all three components is stochastic. As such, the dynamics of the bank value can be described by a general SDE:

$$d\mathbf{VB}(t) = \mu_t^{\mathbf{VB}} \mathbf{VB}(t) dt + \sigma_t^{\mathbf{VB}} \mathbf{VB}(t) dW_t^{\mathbf{VB}}. \tag{2}$$

Assuming that any financial contract depend on the K risk factors and that the cash is invested at the risk-free rate r, which is a variable that depends on the stochastic factors as well,[3] we can further expand Equation (2) by applying Ito's lemma to all assets, cash and liabilities (assuming they are all generalised Ito processes), so that the drift is:

$$\mu_t^{\mathbf{VB}} = \left[\sum_{i=1}^{I} \mu_t^{A_i} + \mu_t^B - \sum_{j=1}^{J} \mu_t^{L_j} \right], \tag{3}$$

where, for a contract $O = \{A, B, L\}$ in the assets, cash or liabilities:

$$\mu_t^O = \left[\sum_{k=1}^{K} \mu_t^k \frac{\partial O}{\partial Y_k} + \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \mathbf{cov}(Y_k, Y_{k_1}) \frac{\partial^2 O}{\partial Y_{k_1} \partial Y_{k_2}} + \frac{\partial O}{\partial t} \right]. \tag{4}$$

[2]Since we consider only the value of the bank to shareholders relevant for incremental valuation of financial contracts, we will refer only to it in the following, even when simply writing "value of the bank" omitting "to shareholders".

[3]Hence, the cash invested in the bank account is a risky asset too. The cash yields an interest proportional to the rate r that is risk-free as far as the dafault risk of the counterparty is concerned, but it does not exclude all other risks.

The diffusion coefficient is:

$$\sigma_t^{\mathbf{VB}} = \left[\sum_{i=1}^{I} \sigma_t^{A_i} + \sigma_t^B - \sum_{j=1}^{J} \sigma_t^{L_j}\right] \frac{1}{\mathbf{VB}(t)}, \tag{5}$$

where, for a contract O in the assets or liabilities:

$$\sigma_t^O = \left[\sum_{k=1}^{K} \mu_t^k \frac{\partial O}{\partial Y_k}\right] \frac{1}{O_t}. \tag{6}$$

Given Theorem 2 in CIR [6], we have that for any contract (contingent claim) or cash $O = \{A, B, L\}$, the equilibrium expected return is:[4]

$$\mu_t^O = r_t O(t) + \sum_{k=1}^{K} \frac{\partial O}{\partial Y_k} \left[\frac{-\frac{\partial^2 \mathcal{J}}{\partial W^2}}{\frac{\partial \mathcal{J}}{\partial W}} \mathbf{cov}(W, Y_k) + \sum_{k_1=1}^{K} \frac{-\frac{\partial^2 \mathcal{J}}{\partial W \partial Y_{k_1}}}{\frac{\partial \mathcal{J}}{\partial W}} \mathbf{cov}(Y_k, Y_{k_1}) \right], \tag{7}$$

where $\mathcal{J}$ is the indirect utility function of the bank, which may clash with the indirect utility function of the representative market agent. Setting

$$\phi_{Y_k} = \left[\frac{-\frac{\partial^2 \mathcal{J}}{\partial W^2}}{\frac{\partial \mathcal{J}}{\partial W}} \mathbf{cov}(W, Y_k) + \sum_{k_1=1}^{K} \frac{-\frac{\partial^2 \mathcal{J}}{\partial W \partial Y_{k_1}}}{\frac{\partial \mathcal{J}}{\partial W}} \mathbf{cov}(Y_k, Y_{k_1}) \right],$$

the drift of an asset or a liability can be written as:

$$\mu_t^O = r_t O(t) + \sum_{k=1}^{K} \frac{\partial O}{\partial Y_k} \phi_{Y_k}. \tag{8}$$

For a contingent claim or cash $O = \{A, B, L\}$, let $\lambda_t^O = \sum_{k=1}^{K} \frac{1}{O_t} \frac{\partial O}{\partial Y_k} \phi_{Y_k}$; by combining Equation (3), (4) and (8) (see also Theorem 3 in CIR [6]), the bank value satisfies the following PDE:[5]

$$\begin{aligned} &\sum_{i=1}^{I} \mu_t^{A_i} + \mu_t^B - \sum_{j=1}^{J} \mu_t^{L_j} = \\ &\qquad r_t \left[\sum_{i=1}^{I} A_i(t) + B(t) - \sum_{j=1}^{J} L_j(t)\right] + \sum_{i=1}^{I} \lambda_t^{A_i} A_i(t) + \lambda_t^B B(t) - \sum_{j=1}^{J} \lambda_t^{L_j} L_j(t). \end{aligned} \tag{9}$$

To be a risk-premium (*i.e.*: an increase of the yield earned from the ownership of the bank and requested by shareholders), $\lambda_t^{A_i} \geq 0$, $\lambda_t^B \geq 0$, and $\lambda_t^{L_j} \leq 0$. The solution to Equation (9) can be written in terms of expectation (see Friedman [7], Theorem 5.2):

$$\begin{aligned} \mathbf{VB}(t) = &\mathbf{E}^P \Bigg[\Bigg[\mathcal{D}(t,T) \mathbf{VB}(T) \\ &- \int_t^T \mathcal{D}(t,s) \left(\sum_{i=1}^{I} \lambda_s^{A_i} A_i(s) + \lambda_s^B B(s) - \sum_{j=1}^{J} \lambda_t^{L_j} L_j(s) \right) ds \Bigg] \mathbf{1}_{\{\tau_B > T\}} \Bigg], \end{aligned} \tag{10}$$

[4] We recall that we assumed that the contingent claims' pay-off depend only on the risk factors Y_k and not the wealth W.

[5] A more explicit way to write Equation (9) is:

$$\begin{aligned} &\sum_{i=1}^{I} \left[\sum_{k=1}^{K} \mu_t^k \frac{\partial A_i}{\partial Y_k} + \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \rho_t^{k_1,k_2} \sigma_t^{k_1} \sigma_t^{k_2} \frac{\partial^2 A_i}{\partial Y_{k_1} \partial Y_{k_2}} + \frac{\partial A_i}{\partial t} \right] \\ &+ \left[\sum_{k=1}^{K} \mu_t^k \frac{\partial B}{\partial Y_k} + \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \rho_t^{k_1,k_2} \sigma_t^{k_1} \sigma_t^{k_2} \frac{\partial^2 B}{\partial Y_{k_1} \partial Y_{k_2}} + \frac{\partial B}{\partial t} \right] \\ &- \sum_{j=1}^{J} \left[\sum_{k=1}^{K} \mu_t^k \frac{\partial L_j}{\partial Y_k} + \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \rho_t^{k_1,k_2} \sigma_t^{k_1} \sigma_t^{k_2} \frac{\partial^2 L_j}{\partial Y_{k_1} \partial Y_{k_2}} + \frac{\partial L_j}{\partial t} \right] \\ &= r_t \left[\sum_{i=1}^{I} A_i(t) + B(t) - \sum_{j=1}^{J} L_j(t) \right] + \sum_{i=1}^{I} \sum_{k=1}^{K} \frac{\partial A_i}{\partial Y_k} \phi_{Y_k} + \sum_{k=1}^{K} \frac{\partial B}{\partial Y_k} \phi_{Y_k} - \sum_{j=1}^{J} \sum_{k=1}^{K} \frac{\partial L_j}{\partial Y_k} \phi_{Y_k}, \end{aligned}$$

which is similar to the PDE in Theorem 3 in CIR [6].

where $\mathcal{D}(t,T) = \exp\left[-\int_t^T r_s ds\right]$ is the discount factor from time t to time T, $\mathbf{VB}(T) = \sum_{i=1}^{I} A_i(T) + B(T) - \sum_{j=1}^{J} L_j(T)$ is the terminal value of the bank's equity,[6] and $\mathbf{1}_{\{\tau_B > T\}}$ is the indicator function equal to 1 until when the banks default occurs. Bank's default is defined ad the first time between the reference time t and the final time T when the assets and cash are smaller than the liabilities:

$$\tau_B = \inf\left\{t \le s \le T : \sum_{i=1}^{I} A_i(s) + B(s) < \sum_{j=1}^{J} L_j(s)\right\}.$$

Some comments are in order: the expectation in Equation (10) is taken under the real world measure P, which means that all the drifts of the risk factors Y_k are those of the real world dynamics in Equation (1). It should be noted that the discounting is operated with the risk free rate r, as it is typically the case when the expectation is taken under the risk neutral measure Q, *i.e.*: when the dynamics of the risk factors are risk neutral, instead of those in Equation (1). To account for the error made in discounting with the risk free rate pay-offs that depend on real world dynamics of the risk factors, we add an adjustment equal to the risk-premia for all risk factors, referring to each contingent claim in the assets and in the liabilities of the bank's balance sheet.

We have now to examine two possible cases when the banks buys an assets, or issue debt.

1.1 The Bank is Price Taker

Let us assume that the bank buys all assets in the market and it has no greater bargaining power than any other agent. In this case, the bank must accept the prices set by the market for all the assets it buys. The bank funds the purchase of the assets by issuing debt claims whose price is also set by the creditors, and it passively has to accept it since it has no bargaining power.

This assumption implies that the drift of each asset is determined by the market with no possibility for the bank to affect it. The same reasoning can be applied also to debt claims in the liabilities. The risk-premium requested by the market for any bank's asset or a liability is

$$\vartheta_t^O = \sum_{k=1}^{K} \frac{1}{O_t} \frac{\partial O}{\partial Y_k} \varphi_{Y_k},$$

for $O = \{A, B, L\}$, and φ_{Y_k} defined as ϕ_{Y_k} except its dependence on the indirect utility function $\mathcal{J}$ of the representative market agent, instead of that of the bank. It should be stressed that, from the perspective of a market agent other than the bank, the latter's debt claims are just assets traded in the market evaluated according to the same criteria as others.

So the drift of any asset, or bank's debt claim, is: $\mu_t^O = r_t O(t) + \vartheta_t^O O_t$, by replacing which in Equation (9) we have:

$$\begin{aligned} &\sum_{i=1}^{I}[r_t + (\vartheta_t^{A_i} - \lambda_t^{A_i})]A_i(t) + [r_t + (\vartheta_t^B - \lambda_t^B)]B(t) - \sum_{j=1}^{J}[r_t + (\vartheta_t^{L_j} - \lambda_t^{L_j})]L_j(t) = \\ &r_t\left[\sum_{i=1}^{I} A_i(t) + B(t) - \sum_{j=1}^{J} L_j(t)\right]. \end{aligned} \tag{11}$$

Equation (11) shows that, under the assumption the bank is price taker, when the market risk-premia $\vartheta_t^{A_i} > \lambda_t^{A_i}$ the expected return on the assets, adjusted for the risk-premium requested by the bank, is above the risk free rate, its inclusion in the balance sheet is positively contributing to the (present) value of the bank (and the reverse is true when $\vartheta_t^{A_i} < \lambda_t^{A_i}$).

For the bank's liabilities, from Equation (9) it is clear that the quantities $\lambda_t^{L_j}$ must have a negative value if they have to contribute as premia and not as penalties over the risk-free rate. This means that $\vartheta_t^{L_j} \ge \lambda_t^{L_j}$ always, since the debt issued by the bank is an asset when purchased by a creditor and he/she will require a positive premium over the risk-free rate ($\vartheta_t^{L_j} \ge 0$). In the end, the risk-premia of the creditor and of the bank are not netting out as for assets, but they are adding up.

[6] The final time T can be taken equal to the longest maturity of the assets and of the liabilities in the bank's balance sheet.

In conclusion, we can affirm that when the bank has to issue debt claims whose price is set by the market, it will pay twice the risk-premium: the risk-premium embedded in the price required by the buyers (creditors of the bank), and the risk-premium that the bank has not been able to include in the price. While for the assets the market's and bank's risk-premia are affecting the total bank value only for the net difference, for the liabilities the difference is actually a sum of two risk-premia (since they must have opposite signs) and as such acting on an aggregated basis on the bank value. Alternatively said, when buying assets, the banks is competing on the same side with other investors/buyers, so that only the differences between the premia required by the bank and those required by the representative market investor affect the bank's total present value; when issuing debt, the bank is on the opposite side with respect to the market investors/buyers: as such, the risk-premia have opposite signs and they do not net out when computing the bank's total present value.

Finally, by assuming that $\vartheta_t^{A_i} = \lambda_t^{A_i}$ and that $\lambda_t^{L_j} = 0$, for any i and j, the expectation in (10) can be calculated by setting the risk neutral drift for all assets, and by considering just the drift implied in market prices for all liabilities.[7]

1.2 The Bank is Price Maker

Let us assume that the bank is able to set the price when buying assets and when issuing new debt. In this circumstances, the price should be such that it embed a risk-premium such that the bank value at least does not decline after the inclusion of the new asset, or new liability, in the bank's balance sheet. From Equation (9) it is clear that the bank must set the premium over the risk free rate for the assets it buys, or the liabilities it issues, equal to its own premium parameters $\lambda_t^{A_i}$ and $\lambda_t^{L_j}$.

It is also clear now that the bank should set a price so that its expected return (*i.e.*: the drift) is lower than risk free rate when issuing liabilities $\lambda_t^{L_j}$. This is what actually happens in reality when the bank has a strong bargaining power, as it is the case for example with retail depositors, when the competition with other banks is not fierce.

If the bank is able to set the premia on all assets and liabilities equal to its own levels, than the evaluation of the bank's value reduces to the standard risk-neutral one. More specifically, the drifts in Equation (9) will be equal to risk fee rate, as well as the return on the net value of the bank (*i.e.*: the equity, the right-hand side of Equation (9)). Equation (10) can be rewritten as:

$$\mathbf{VB}(t) = \mathbf{E}^{\mathbb{Q}}\left[\mathcal{D}(t,T)\mathbf{VB}(T)\mathbf{1}_{\{\tau_B > T\}}\right]. \quad (12)$$

We have to stress the fact that, in order to calculate correctly the value of the bank, one should use the real world dynamics to verify the possibility of the bank's default at any time $t < s \le T$ ($\tau_B = s$). This means that in any case it is still necessary to know the physical measure drifts of all assets and liabilities; moreover, a true simplification of formula (10) is only possible by checking for the bank's default by using risk neutral drifts for all contracts in the balance sheet.[8] This case will produce a bank's present value that is not strictly consistent with the general equilibrium evaluation framework that we have adapted from CIR [6].

It is worth noting that in the general equilibrium framework above, we have identified the conditions under which risk-neutral measures can be used and we have also justified the discounting with the risk-free rate, after properly adjusting the drifts. A different approach has been followed by Andersen *et al.* [1], who mention rather quickly, and without analysing the hidden assumptions, the Stiemke's Lemma, which together with some coherency assumptions justifies the classical risk-neutral valuation.[9]. We think the general equilibrium approach allows to come up with a valuation framework that is sounder and richer, with implications not explored in the above mentioned and other works, which we will study in the following.

[7] It is maybe worth noting that the price of the bank's liabilities, when set by the market, is in the end the interest rate requested on the bank's debt: the market risk-premium is included in this quantity.

[8] The same considerations apply also to the first case when the bank is a price taker.

[9] Similarly, Kjaer [10] recurs to the Riesz's representation theorem to justify the risk-neutral valuation of the dealer (bank) balance sheet.

1.3 An Interpretation of the KVA

The Capital Value Adjustment (**KVA**) is the most recent item of the list of adjustments to the "pure" value of a contract, and it has been analysed by several authors: for an excellent review of the matter, and the regulatory and managerial concerns that originate the need for such adjustment, we refer to Prampolini and Morini [11] and the bibliography therein, which contains all the relevant literature at the time of writing.

The term in the integral in the second part of Equation (10) (*i.e.*: $\sum_{i=1}^{I} \lambda_s^{A_i} A_i(s) + \lambda_s^B B(s) - \sum_{j=1}^{J} \lambda_t^{L_j} L_j(s)$) can be seen as the risk-premium that is earned by the equity capital. Actually, from the balance sheet equivalence, at any time t the equity capital $E(t) = \sum_i A_i(t) + B(t) - \sum_j L_j(t)$: or, the equity capital is the liquidation value of the bank if all assets are sold and all liabilities bought back in t, without considering the value originated by seeing the bank as an on-going business activity with a limited liability of the shareholders in case of default. So, we can write the risk-premium of the equity capital π as a weighted average of the risk-premia earned by assets, cash and liabilities:

$$\pi(t) = \left(\sum_{i=1}^{I} \lambda_s^{A_i} \frac{A_i(s)}{E(s)} + \lambda_s^B \frac{B(s)}{E(s)} - \sum_{j=1}^{J} \lambda_t^{L_j} \frac{L_j(s)}{E(s)} \right).$$

By replacing this definition of the π and using the balance sheet equivalence, Equation (10) can be written as:

$$\mathbf{VB}(t) = \mathbf{E}^P \left[\left[\mathcal{D}(t,T)\mathbf{VB}(T) - \int_t^T \mathcal{D}(t,s)\pi(s)E(s)ds \right] \mathbf{1}_{\{\tau_B > T\}} \right]. \quad (13)$$

Let us assume that the bank determines the structure of the investments, and of the liabilities used to finance them, so that at any time t it has enough equity capital $E(t)$ to bear potential losses L occurring according to a given (real-world) distribution. The level of equity capital that is needed to that end is usually termed Economic Capital (**EC**) and it can be seen as the level that avoids the default of the bank with a given level of confidence $1-q$, so that the (real-world) default probability is q. We can formally define **EC** as:

$$\begin{aligned} \mathbf{EC}(t) &= \inf\{E(t) : \mathbf{P}^P(\tau_b \le t) = 1 - q\} \\ &\inf\left\{ E(t) : \mathbf{P}^P\left(\sum_{i=1}^{I} \lambda_s^{A_i} A_i(s) + \lambda_s^B B(s) - \sum_{j=1}^{J} \lambda_t^{L_j} L_j(s) \le 0 \right) = q \right\}. \end{aligned} \quad (14)$$

It should be noted that the losses of the bank that can be borne by the bank with a probability $1-q$ are clashing with the definition of Value-at-Risk of a portfolio of contracts equal to the bank's balance sheet, as we will see in a while.

Typically, the bank calculates $\mathbf{EC}(t)$ at the reference time 0, in terms of losses $L = \min[E(t) - E(0), 0]$, where $E(s)$ is the equity capital at time s. Let the expected variation of the equity be $\Delta E(t) = \mathbf{E}[E(t) - E(0)]$ (which could also be positive, *i.e.*: a profit): the expected equity capital in t can be written as:

$$\mathbf{E}[E(t)] = E(0) + \Delta E(t).$$

When the $\mathbf{EC}(t)$ is determined at time 0, the bank wishes that the expected equity capital is such that $\mathbf{E}[E(t)]$ is able to recover expected variations and unexpected losses, or:

$$\mathbf{E}[E(t)] = [\mathbf{VaR}_{1-q}(L) - \Delta E(t)] + \Delta E(t) = \mathbf{EC}(t),$$

where $\mathbf{VaR}_{1-q}(L)$ is the loss level at a given confidence level,[10] defined as:

$$\mathbf{VaR}_{1-q}(L) = \inf\{l : \mathbf{P}^P(L \ge l) \le q\}.$$

[10] This level is the well known VaR of a portfolio, or of the bank's equity capital in the present case.

The Economic Capital at time t implies that the capital $K_q(0)$ that must be allocated at time 0, is $K_q(0) = [\mathbf{VaR}_{1-q}(L) - \Delta E(t)]$: this is how typically banks operate in the real world.[11]

Assuming that the Economic Capital is always allocated one period before the date it refers to (*e.g.*: 1 year), the **KVA** can be easily defined as:

$$\mathbf{KVA}(t,T) = \mathbf{E}^P\left[\left(\int_t^T \mathcal{D}(t,s)\pi(s)K_q(s)ds\right)\mathbf{1}_{\{\tau_B>T\}}\right]. \tag{15}$$

It is the adjustment in the evaluation formula (10) when the equity capital is set in such a way that it matches the Economic Capital as defined above. Some considerations are in order:

- The definition of Economic Capital given above can be applied both to risk-based measure (*e.g.*: simulation models applied to the bank's balance sheet) and non-risk-based measures (*e.g.:* regulatory formulae): for a discussion of both types of measures, see Prampolini and Morini [11].

- The **KVA** is consistently computed only under the real-world measure P and it is discounted with the risk-free rate: these are not assumptions or choices arbitrarily made, but both are naturally derived from the framework sketched above (different discount factors can be found in Prampolini and Morini [11], Kjaer [10], Brigo *et al.*[2], Green *et al.* [8]: in some cases the discount factors include the intensity of default of the counterparty and of the bank, in any case they are not consistently derived within an equilibrium framework such as the one above). When one wants to compute the bank's value under the risk-neutral measure, the inclusion of the **KVA** is not consistent, unless the adjustment includes only the difference between the bank's and the market's risk-premia, in which case Equation (10) becomes:

$$\begin{aligned}\mathbf{VB}(t) =& \mathbf{E}^Q\Bigg[\Bigg[\mathcal{D}(t,T)\mathbf{VB}(T)\\ &- \int_t^T \mathcal{D}(t,s)\Bigg(\sum_{i=1}^{I}(\lambda_s^{A_i} - \vartheta_s^{A_i})A_i(s) + (\lambda_s^B - \vartheta_s^B)B(s)\\ &- \sum_{j=1}^{J}(\lambda_s^{L_j} - \vartheta_s^{L_j})L_j(s)\Bigg)ds\Bigg]\mathbf{1}_{\{\tau_B>T\}}\Bigg]\\ =& \mathbf{E}^Q\left[\mathcal{D}(t,T)\mathbf{VB}(T)\mathbf{1}_{\{\tau_B>T\}}\right] - \overline{\mathbf{KVA}}(t,T),\end{aligned} \tag{16}$$

  where
  $\overline{\mathbf{KVA}}(t,T) = \mathbf{E}^Q\left[\left(\sum_{i=1}^{I}(\lambda_s^{A_i} - \vartheta_s^{A_i})A_i(s) + (\lambda_s^B - \vartheta_s^B)B(s) - \sum_{j=1}^{J}(\lambda_s^{L_j} - \vartheta_s^{L_j})L_j(s)\right)ds\right)\mathbf{1}_{\{\tau_B>T\}}\right] =$
  $\mathbf{E}^P\left[\left(\int_t^T \mathcal{D}(t,s)\pi^*(s)K_q(s)ds\right)\mathbf{1}_{\{\tau_B>T\}}\right]$ and $\pi^*(t)$ is accordingly defined as $\pi(t)$.

- The remuneration of the Economic Capital is given by only the risk-premia embedded in the assets, cash (bank account) and liabilities, either set by the market or by the bank depending on the bank's bargaining power in each case. This result in in striking contrast with all the literature publicly available at the time of writing (see the point above), where the remuneration encompasses the entire return on the contracts in the balance sheet. This will produce a double counting of the risk-free rate within the calculation of the bank's value, which will also imply a wrong adjustment if the bank is able to set a contract's price. Our result descends from the general equilibrium framework and is valid in the case the value is computed under the real-world measure, otherwise the inclusion of the **KVA** adjustment is quite untenabale.

[11]It should also be added that in the real world, banks do not jointly compute the VaR of the entire balance sheet, as implicit in Equation (14), but they rather compute several types of VaR, each one referring to a specific risk (*e.g.*: market, credit, operational, etc.), with their *ex-post* aggregation with some techniques that try to account for the over estimation produced by the simple summation (*e.g.*: aggregation via copula functions). The approach is not theoretically optimal, but it is also one of the practical solutions to overcome the burdensome alternative to calculate the true aggregated VaR as in equation (14).

- The risk-premium of the equity capital is a weighted average of the risk-premia of the different items of the balance-sheet: when the bank has pricing power, it can require a premium proportional to the risk of the contract and the incremental Economic Capital needed to preserve the same probability of default of the bank. Pricing based on RAROC criteria are common choices, lately suggested also in Prampolini and Morini [11] and Brigo *et al*.[2].

2. A Non-Trivial Set-Up to Evaluate Contracts

We can specify the general framework sketched above to evaluate the incremental contribution of a contingent claim in the balance sheet of the bank. First, we outline how to calculate the value of the bank; then, we will assess how the insertion of a new contract in the bank's balance sheet impacts the value.

2.1 The Bank Value at the Reference Date

Let us set the evaluation reference date in $t = 0$ and the terminal date in T, which is equal to, or greater than, the expiry of the longest maturing contract in the bank's balance sheet. The bank trades with $j = 1, ..., J$ counterparties: so there are J netting sets. Each netting set n contains $i_n = 1, ..., I_n$ contracts, V_{i_n}. The "pure" value[12] of the netting set is $V_n = \sum_{i=1}^{I} = V_{i_n}$. Netting sets can be collateralised with collateral C_n.

Assets' and liabilities' cash-flows $\mathbf{CF}(t_m)$ are deposited in, or withdrawn from, the bank account B that instantaneously earns the risk-free rate r and that has supposed to be counterparty risk-free. Assuming that the initial amount $B(0) \geq 0$, the present value $\mathcal{B}(0,t)$ in 0 of the bank account considered up to time s is:

$$\mathcal{B}(0,t) = B(0) + \sum_{m=1}^{M} \mathbf{CF}(t_m) \int_0^s \mathcal{D}(0,t_m)\delta(t_m - v)dv + \int_0^s \mathcal{D}(0,v)B(v)r_v dv. \tag{17}$$

Equation (18) states that the present value of the bank account is simply the sum of the starting value of the account ($B(0)$), the present value of all the cash-flows originated by the assets and the liabilities ($\delta(x)$ is the Dirac function centered in x). Since all cash-flows are evaluated also within the contracts in the balance sheet, to avoid any double counting, for the evaluation of the bank at the reference date we consider a modified version of the present value of the bank account, which exclude all cash-flows generated by the assets and the liabilities from a given time t to the terminal date T:

$$\mathcal{B}^*(t,T) = \mathcal{B}(0,t) + \int_t^T \mathcal{D}(0,v)B(v)r_v dv. \tag{18}$$

The bank account can never be below 0, so when cumulative cash-flows of existing contracts imply a negative balance on the bank account, short-term debt (liabilities) are issued by the bank:

$$SL(t) = |\min[B(t), 0]|.$$

The present value of the short-term debt in 0 up to T is:

$$\mathcal{S}(0,T) = SL(T)D(0,T) + \int_0^t \mathcal{D}(0,s)SL(s)[r_s + s_s^B]ds, \tag{19}$$

where $SL(t)$ is the notional of the short-term debt outstanding in t and s^B is the funding spread paid by the bank over the risk free rate. The short-term can be repaid, in full or partially, whenever at some time t, the bank account $B(t) > 0$.

At time 0 the bank has also some long-term debt outstanding LL, expiring in $T_L \leq T$. We assume the long term debt is rolled over for a period equal to T_L, or up to T if the renewed debt outlives the evaluation horizon. The present value of the long-term debt in 0 up to the evaluation horizon T is:

$$\mathcal{L}(0,t) = \sum_{n=1}^{N} \kappa(t_n)LL \int_0^T \mathcal{D}(0,t_n)\delta(t_n - s)ds + \mathcal{D}(0,T)LL, \tag{20}$$

[12] By "pure value" we mean the value calculated in a world with not credit risk and, hence, collateral agreements.

where $\kappa(t_n)$ is the coupon paid on the notional LL, and N is the number of coupons to pay between 0 and $t_N = T$.

We allow also for the possibility that contracts withe counterparty j are collateralised, according to some rules, in cash, thus generating a collateral account equal to $C(t)$ in t. A positive balance ($C(t) > 0$) means that the bank must pay the collateral rate c to the counterparty j, but at the same time can invest the amount at the market risk-free rate r. The net cost (or gain, if the balance in negative ($C(t) > 0$) is then:

$$\mathcal{C}_j(0,t) = \int_0^T \mathcal{D}(0,s)C_j(s)(r_s - c_s)ds. \tag{21}$$

We are simplifying the notation by assuming that the collateral rate is the same for all counterparties. It should be noted that the account balance $C(t)$ is deposited in (or withdrawn from) the bank cash account $B(t)$.

To make things more tractable, we assume that the joint default of two counterparties k and h is zero, so that:

$$P(\tau_k = \tau_h) = 0,$$

where τ_j is the default time of the counterparty j. When the counterparty j defaults, the value of the bank that includes also the loss given default is defined as $\mathbf{VB}(\tau_j)$.

Considering the defaults of all the counterparties, and setting $\mathbf{VB}^*(t) = \mathbf{VB}(t) - \mathbf{KVA}(t,T)$, the value of the bank in 0 can be written as:

$$\begin{aligned}\mathbf{VB}(0) = \mathbf{E}^P\Big[\Big[\sum_j \mathcal{D}(0,T)\mathbf{VB}^*(T)(1 - \mathbf{1}_{\{\tau_j<T\}}) + \\ \sum_j \max\left[\mathcal{D}(0,\tau_j)\mathbf{VB}^*(\tau_j), 0\right]\mathbf{1}_{\{\tau_j<T\}}\Big]\mathbf{1}_{\{\tau_B>T\}}\Big].\end{aligned} \tag{22}$$

We stress the fact that the measure, under which the expectation is calculated, is the real-world measure. Equation (22) states that the bank value is equal to the expected discounted terminal value, provided that no default of the counterparties occurs (first line), or the remaining value left after the default of the counterpaarties floored at 0 (second line), provided that the bank does not go bust itself (the indicator function of the bank's default multiplying the entire value within the expectation operator). Equation (22) can be equivalently written as:

$$\begin{aligned}\mathbf{VB}(0) = \mathbf{E}^P\Big[\Big[\mathcal{D}(0,T)\mathbf{VB}^*(T) - \\ [\sum_j \mathcal{D}(0,T)\mathbf{VB}^*(T) - \max\left[\mathcal{D}(0,\tau_j)\mathbf{VB}^*(\tau_j), 0\right]\mathbf{1}_{\{\tau_j<T\}}]\Big]\mathbf{1}_{\{\tau_B>T\}}\Big],\end{aligned} \tag{23}$$

which is simply the expected present value of the bank at expiry T, deducted the expected loss generated by the default of counterparty j (second line), provided that the bank survives until time T.

Equation (23) can be written more explicitly by considering the contract the sum of the contracts with the counterparty j, whose net value in T is $V_j(0,T)$, the bank account, the collateral account net cost, and the short-term and long-term liabilities defined above:

$$\begin{aligned}\mathbf{VB}(0) = \mathbf{E}^P\Big[\Big[\sum_j \mathcal{D}(0,T)V_j(0,T) + \mathcal{C}_j(0,T) + \mathcal{B}^*(0,T) - [\mathcal{S}(0,T) + \mathcal{L}(0,T)] - \mathbf{KVA}(0,T) \\ - \sum_j \Big[\mathcal{D}(0,T)V_j(\tau_j,T) + \mathcal{C}_j(0,\tau_j) + \mathcal{B}^*(\tau_j,T) - [\mathcal{S}(0,\tau_j) + \mathcal{L}(\tau_j,T) - \mathbf{KVA}(\tau_j,T)] \\ - \max\big[\sum_{k\neq j} \mathcal{D}(0,\tau_j)\big(V_k(\tau_j,T) + \mathbf{Rec}_j V_j^+(\tau_j,T) + V_j^-(\tau_j,T)\big) \\ + \mathcal{C}_k(0,\tau_j) + \mathcal{B}^*(\tau_j,T) - [\mathcal{S}(0,\tau_j) + \mathcal{L}(\tau_j,T) - \mathbf{KVA}(\tau_j,T)], 0\big]\Big]\mathbf{1}_{\{\tau_j<T\}}\Big]\mathbf{1}_{\{\tau_B>T\}}\Big],\end{aligned} \tag{24}$$

where V_j^+ is the net value of the contracts with counterparty j, if positive, and $\mathbf{Rec}_j$ is the percentage recovered after the default of the latter; V_j^- is the net value of the contracts if negative. Basically, Equation restates, in more specific terms, that the bank value is the expected discounted terminal value, deducted the expected losses given the default of the counterparty j, provided that the bank survives until T.

Equation (2.1) can be made more readable if we define the aggergated xVAs of the bank's balance sheet. Let us start with the Credit Value Adjustment (**CVA**) for the netting set of contracts referring to counterparty j, which takes into account the limited liability of the bank's shareholders, defined as:

$$\mathbf{CVA}_j^{LL}(0,T) = \mathbf{E}^P\Bigg[\Bigg[\Big[\mathcal{D}(0,T)V_j(\tau_j,T) + \mathcal{C}_j(0,\tau_j) + \mathcal{B}^*(\tau_j,T) - [\mathcal{S}(0,\tau_j) + \mathcal{L}(\tau_j,T) - \mathbf{KVA}(\tau_j,T)] \\ - \max\big[\sum_{k\neq j}\mathcal{D}(0,\tau_j)\big(V_k(\tau_j,T) + \mathbf{Rec}_j V_j^+(\tau_j,T) + V_j^-(\tau_j,T)\big) \\ + \mathcal{C}_k(0,\tau_j) + \mathcal{B}^*(\tau_j,T) - [\mathcal{S}(0,\tau_j) + \mathcal{L}(\tau_j,T) - \mathbf{KVA}(\tau_j,T)], 0\big]\Big]\mathbf{1}_{\{\tau_j<T\}}\Bigg]\mathbf{1}_{\{\tau_B>T\}}\Bigg]. \tag{25}$$

The liquidity value adjustment $\mathbf{LVA}_j^{LL}$, referring to the collateral agreement with counterparty j, accounting for the bank's default, is:

$$\mathbf{LVA}_j^{LL}(0,T) = \mathbf{E}^P\Bigg[\Bigg[\int_0^T \mathcal{D}(0,s)C_j(s)(r_s - c_s)ds\Bigg]\mathbf{1}_{\{\tau_B>T\}}\Bigg]. \tag{26}$$

For The Funding Value Adjustment $\mathbf{FVA}^{LL}$, first we need to define the risk-free equivalent of the short-term $\mathcal{S}^*(0,T)$ and $\mathcal{L}^*(0,T)$ and long-term debts, which means that they earn an interest equal to the short-term and the long-term risk-free, respectively, without any funding spread. Formally, we have:

$$\mathcal{S}(0,T) = SL(T)D(0,T) + \int_0^t \mathcal{D}(0,s)SL(s)r_s ds, \tag{27}$$

and

$$\mathcal{L}^*(0,T) = \sum_{n=1}^{N}\kappa^*(t_n)LL\int_0^T \mathcal{D}(0,t_n)\delta(t_n - s)ds + \mathcal{D}(0,T)LL, \tag{28}$$

where $\kappa^*(t_n)$ is the coupon paid on the notional LL of a debt expiring in T, based on the market risk-free interest rate curve. The **FVA** accounting for the bank's default, is:[13]

$$\mathbf{FVA}^{LL}(0,T) = \mathbf{E}^P\Bigg[\Bigg[\int_0^T \mathcal{D}(0,s)SL(s)s_s^B ds + [\mathcal{L}^*(0,T) - \mathcal{L}(0,T)]\Bigg]\mathbf{1}_{\{\tau_B>T\}}\Bigg]. \tag{29}$$

Finally, the Capital Value Adjustment $\mathbf{KVA}^{LL} = \mathbf{KVA}$ is defined in Equation (15).

Inserting the xVAs in the valuation formula (2.1), the value of the bank can be written as follows:

$$\mathbf{VB}(0) = \mathbf{E}^P\Bigg[\Bigg[\sum_j \mathcal{D}(0,T)V_j(0,T) + \mathcal{C}_j(0,T) + \mathcal{B}^*(0,T) - [\mathcal{S}^*(0,T) + \mathcal{L}^*(0,T)]\Bigg]\mathbf{1}_{\{\tau_B>T\}}\Bigg] \\ - \mathbf{KVA}^{LL}(0,T) - \mathbf{FVA}^{LL}(0,T) + \sum_j\big[\mathbf{LVA}_j^{LL}(0,T) - \mathbf{CVA}_j^{LL}(0,T)\big], \tag{30}$$

or alternatively:

$$\mathbf{VB}(0) = \mathbf{E}^Q\Bigg[\Bigg[\sum_j \mathcal{D}(0,T)V_j(0,T) + \mathcal{C}_j(0,T) + \mathcal{B}^*(0,T) - [\mathcal{S}^*(0,T) + \mathcal{L}^*(0,T)]\Bigg]\mathbf{1}_{\{\tau_B>T\}}\Bigg] \\ - \mathbf{FVA}^{LL}(0,T) + \sum_j\big[\mathbf{LVA}_j^{LL}(0,T) - \mathbf{CVA}_j^{LL}(0,T)\big], \tag{31}$$

[13] The **FVA**, and the bank vale **VB**, would be slightly more complicated if the long term debt were not issued at 0, as we are assuming in this framework. In this case, we need to adjust the difference between the values of the risk-free equivalent and the actual debts to account for the effects due to the possible changes in the risk-free market rates, which affect the discount

where no difference is assumed to exist between market and bank's risk-premia.

It is worth noting that all the xVA include the limited liability of the shareholders, since the indicator function of the bank's default is present. This means that there is no need to include in the value of the bank in (30) the Limited Liability Value adjustment that was introduced in Castagna [5]. In that case the adjustment was justified by the fact that the other xVA did not account for the shareholders' limited liability and the bank's default.

3. Incremental Valuation of a New Contract

Assume at time $t > 0$ a new contract $I+1$ of the $J-$th netting set, V_{I+1_J}, is included in the bank's balance sheet; let us assume that the expiry of the contract matches the terminal evaluation date T (even if it will not be the case in most of cases). Let $\mathbf{VB}^{+}(t)$ be the value of the bank that includes the new contract. From Equation (30), or (31), the variation of the bank's value $\Delta\mathbf{VB}(t) = \mathbf{VB}^{+}(t) - \mathbf{VB}(t)$ can be decomposed in the following parts:

$$\begin{aligned}\Delta\mathbf{VB}(t) =& \mathbf{E}^{P}\left[V_{I+1_J}(T)\mathbf{1}_{\{\tau_B>T\}} + \Delta B(t)\right] - \Delta\mathbf{KVA}^{LL}(t,T) - \Delta\mathbf{FVA}^{LL}(t,T) \\ &+ \Delta\mathbf{LVA}_J^{LL}(t,T) - \Delta\mathbf{CVA}_J^{LL}(t,T),\end{aligned} \tag{32}$$

or

$$\begin{aligned}\Delta\mathbf{VB}(t) =& \mathbf{E}^{Q}\left[V_{I+1_J}(T)\mathbf{1}_{\{\tau_B>T\}} + \Delta B(t)\right] - \Delta\mathbf{FVA}^{LL}(t,T) \\ &+ \Delta\mathbf{LVA}_J^{LL}(t,T) - \Delta\mathbf{CVA}_J^{LL}(t,T).\end{aligned} \tag{33}$$

The variation of the bank account $\Delta B(t)$ is due to the fact that the bank could finance the possible payment of a cash premium by reducing a positive cash balance; alternatively, the bank can finance the payment of the premium by issuing new short-term or long-term debt, which will affect the **FVA**.

The indifference value to the bank of the new contract is that one that makes the variation of the bank's value nil ($\Delta\mathbf{VB}(t) = 0$): not always the bank is able to trade at the indifference price level, and hence the inclusion of a contract in the balance sheet may have a positive or a negative effect on the bank's value.

One may wonder if Formulae (32) and (33) are equivalent and they can be used indifferently when operating the incremental valuation of a contract. The answer is that, theoretically speaking, the two Formulae are equivalent, but in practice they will be used depending on the cases, the two most common occurrences of which we analyse below:

- The new contract is the purchase of a traded security or a listed derivative: the risk-premium is implicit in the market price, and the bank is a price taker. The incremental valuation can be operated with Formula (33) if the market risk-premium is equal to the bank's risk-premium for that security. Alternatively, Formula (32) can be used, where the real world measure implies an expected return of the security given by the risk-free rate plus the market risk-premium; the positive or negative effect on the value of the bank is produced by the difference between the risk-premium required by the market and by the bank (see Section 1).

- The new contract is an OTC derivative dealt with a weaker counterparty: the bank can exert its bargain power and it can set the value (or traded price, in this case) at the indifference level ($\mathbf{VB}(0) = 0$). To this end, the bank includes all the adjustments and it uses Formula (32): if the contract provides for an initial cash-flow, such as a premium for an option, then since $\mathbf{VB}(0) = 0$, the variation of the bank account will be equal, with the opposite sign, to the variation of the "pure" fair-value of the contract V_j plus the other adjustments:

$$\begin{aligned}\Delta B(t) = & -\mathbf{E}^{P}\left[V_{I+1_J}(T)\mathbf{1}_{\{\tau_B>T\}}\right] + \Delta\mathbf{KVA}^{LL}(t,T) + \Delta\mathbf{FVA}^{LL}(t,T) \\ &- \Delta\mathbf{LVA}_J^{LL}(t,T) + \Delta\mathbf{CVA}_J^{LL}(t,T).\end{aligned} \tag{34}$$

Since $\Delta B(t) = -V_{I+1_J}(t)$, we have that:

$$V_{I+1_J}(t) = \mathbf{E}^P\left[V_{I+1_J}(T)\mathbf{1}_{\{\tau_B>T\}}\right] - \Delta\mathbf{KVA}^{LL}(t,T) - \Delta\mathbf{FVA}^{LL}(t,T) + \Delta\mathbf{LVA}_J^{LL}(t,T) - \Delta\mathbf{CVA}_J^{LL}(t,T), \quad (35)$$

so that the initial value of the contract to the bank can be consistently derived.

As we have already stressed in Section 1, the **KVA** is consistently included in the incremental valuation only if this is operated in the real world measure, so that Formula (33) cannot be used in this case. In case the bank wants to evaluate the contract under the risk neutral measure, it should be either assumed that the market risk-premia are equal to risk-premia required by the bank for the relevant risk factors, or that the **KVA** adjustment should be calculated by considering only the differentials between the market and bank's risk-premia, as in Equation (16) (*i.e.*: by replacing **KVA** with $\overline{\mathbf{KVA}}$).

If the contract does not provide for any initial payment, such as in the case of a swap or a forward, then the "pure" fair value of the contract should be made equal to the negative of the xVAs. Formally, keeping in mind that the bank aims at an indifference value level ($\mathbf{VB}(0) = 0$) and that no cash is paid or received ($\Delta B(t) = 0$), we have that:

$$\mathbf{E}^P\left[V_{I+1_J}(T)\mathbf{1}_{\{\tau_B>T\}}\right] = \Delta\mathbf{KVA}^{LL}(t,T) + \Delta\mathbf{FVA}^{LL}(t,T) - \Delta\mathbf{LVA}_J^{LL}(t,T) + \Delta\mathbf{CVA}_J^{LL}(t,T). \quad (36)$$

A starting value different from zero is achieved by modifying the relevant contract terms, *e.g.*: the forward price level, the swap fixed rate or the spread over the swap floating rate. If the bank manages to close the contract on these terms, then the initial incremental value will be $V_{I+1_J}(t) = 0$.

To recapitulate, basically one should always perform the evaluation under the real world measure if he/she wants to include in a consistent way the **KVA** in the incremental value of the contract. When the evaluator wishes to work under the risk neutral measure, he/she can include the **KVA** under the provision that the bank's risk-premia are taken only differentially with respect to the market risk-premia. Clearly, this means that the current market practice, in most of cases and as it appears from the working papers by practitioners of the financial industry, is over-estimating the impact of the **KVA**. It is also worth stressing that the contribution of the **KVA** to the incremental value of the contract is positive only if the market risk-premia are (generally) higher than those required by the bank, which makes perfect sense. Clearly, if the bank is able to set the price in the bargaining process, it should add the differential **KVA** when going short the contract, and deduct it it when going long. If the bank is a price taker, than the actual contribution of the **KVA** has to be ascertained by comparing the market (or counterparty's) and bank's risk-premia.

Some simplifications can be made in the incremental evaluation of a contract if it has a negligible effect on the total net value of the bank. As seen in Castagna [5], in this case the probability of default of the bank does not change in a meaningful fashion and its variation can be neglected. In practical terms, this means that we have the following: the "pure" value of the contract at expiry can be computed as:

$$\mathbf{E}^P\left[V_{I+1_J}(T)\mathbf{1}_{\{\tau_B>T\}}\right] = \mathbf{E}^P\left[V_{I+1_J}(T)\right]. \quad (37)$$

Similarly, all the variations in the xVAs can be calculated for the each metric without reference to the bank's default:

$$\begin{aligned}\mathbf{CVA}_j^{LL}(0,T) =&\mathbf{CVA}_j(0,T) = \mathbf{E}^P\Big[\Big[\mathcal{D}(0,T)V_j(\tau_j,T) + \mathcal{C}_j(0,\tau_j) + \mathcal{B}^*(\tau_j,T)\\ &- [\mathcal{S}(0,\tau_j) + \mathcal{L}(\tau_j,T) - \mathbf{KVA}(\tau_j,T)]\\ &- \max\Big[\sum_{k\neq j}\mathcal{D}(0,\tau_j)\big(V_k(\tau_j,T) + \mathbf{Rec}_j V_j^+(\tau_j,T) + V_j^-(\tau_j,T)\big)\\ &+ \mathcal{C}_k(0,\tau_j) + \mathcal{B}^*(\tau_j,T) - [\mathcal{S}(0,\tau_j) + \mathcal{L}(\tau_j,T) - \mathbf{KVA}(\tau_j,T)],0\Big]\Big]\mathbf{1}_{\{\tau_j<T\}}\Big].\end{aligned} \tag{38}$$

$$\mathbf{LVA}_j^{LL}(0,T) = \mathbf{LVA}_j(0,T) = \mathbf{E}^P\left[\int_0^T \mathcal{D}(0,s)C_j(s)(r_s - c_s)ds\right]. \tag{39}$$

$$\mathbf{FVA}^{LL}(0,T) = \mathbf{FVA}(0,T) = \mathbf{E}^P\left[\int_0^T \mathcal{D}(0,s)SL(s)s_s^B ds + [\mathcal{L}^*(0,T) - \mathcal{L}(0,T)]\right]. \tag{40}$$

So, the incremental value at time t is:

$$\begin{aligned}V^*_{I+1_J}(t) =&\mathbf{E}^P\Big[V_{I+1_J}(T)\Big] - \Delta\mathbf{KVA}(t,T) - \Delta\mathbf{FVA}(t,T)\\ &+ \Delta\mathbf{LVA}_J(t,T) - \Delta\mathbf{CVA}_J(t,T),\end{aligned} \tag{41}$$

or, under the risk-neutral measure:

$$\begin{aligned}V^*_{I+1_J}(t) =&\mathbf{E}^Q\Big[V_{I+1_J}(T)\Big] - \Delta\overline{\mathbf{KVA}}(t,T) - \Delta\mathbf{FVA}(t,T)\\ &+ \Delta\mathbf{LVA}_J(t,T) - \Delta\mathbf{CVA}_J(t,T).\end{aligned} \tag{42}$$

Equation (42) is formally very similar to the by now standard stand-alone formula used to add the xVAs to the fair value of a contract, with the differences we have already stressed above referring to the **KVA** (which enters only differentially in the evaluation), and without any adjustment due to the Debit Value Adjustment (**DVA**).

The difference between the accurate incremental value and the approximate incremental value has been named in Castagna [5] Limited Liability Value Adjustment **LLVA** and it is formally defined as:

$$\mathbf{LLVA}(t,T) = V^*_{I+1_J}(t) - V_{I+1_J}(t). \tag{43}$$

If the new contract inserted in the balance sheet has a negligible effect on the probability of default of the bank, then the **LLVA** will be very small and it can be disregarded in calculating the incremental value. For contracts with a material impact on the balance sheet and on the bank's default probability, the **LLVA** will be large enough to be included in the incremental valuation. Its effect is to abate the value of a contract that is a liability to the bank (*i.e.*: to make it smaller in absolute terms), and, conversely, increment the value of a contact that is an asset (*i.e.*: to make it greater in absolute terms), similarly to the **DVA**. For a more in-depth discussion on the analogies and the relation of the **LLVA** with the **DVA**, we refer to Castagna [5].

4. Reconciliation with the Modigliani&Miller Theorem

Elsewhere,[14] we had to opportunity to stress that the incremental valuation framework that we introduced above is not in contrast with the main tenet of the Modigliani&Miller (MM) theorem, expounded by the two authors in their article of 1958 (see [9]).[15] On the contrary, when evaluating an investment that is included within the balance sheet of a company (bank) that has already started its operations, then the only way to keep the total value of the assets of the company equal to the

[14]See Castagna [5].

[15]We would like to recall here the MM theorem proves that the value of a project is independent from the way it is financed, or from the capital strucuture of the company undertaking it.

total value of the liabilities, is to apply the principles of the incremental valuation stated in Castagna [5] to the non trivial framework sketched above.

In the recent work by Andersen *et al.* [1], the Modigliani&Miller theorem is proved to be correct when calculating the incremental value of a contract with respect to the total firm value, which is equal to the total value of the assets. In this case, the authors prove that the correct incremental value is given by the “pure” value, deducted of the **CVA** and incremented by the **DVA**, and it is independent from the way it is financed.

In our framework, we calculate the value of a contract only with respect to the value of the bank to the shareholders, because we think this is the only meaningful way the indifference to inclusion of the contract in the balance sheet to all stakeholders. When considering the total value of the firm, the evaluator allows for wealth transfers from shareholders to claimants of higher order, such as bondholders (see Andersen *et al.* [1], pag. 159). On the contrary, when considering the incremental value with respect to the shareholders’ bank value, no wealth transfer is allowed and the contract value is such that all claimants are indifferent to it. Sure, such a value entails additional costs that have to be paid by the counterparty, but here we enter in the market action, where the price of the contract is determined. The price can be set at a level that matches the internal incremental valuation of the bank, thus generating a nil net contribution to the bank value; or it can be different, with a net positive or negative contribution. In any case, the price setting is the result of the bargaining process where the strengths of the bank and of the counterparty clashes and, possibly, they eventually agree to close the deal.

We think that the approach that we have detailed above and that relies on the simpler, but in any case complete, setting in Castagna [5], is in line with Proposition III of Modigliani and Miller [9], where the the optimal investment rule is derived: basically, when the firm (*i.e.*: the manager) is acting in the shareholders’ best interest, it will undertake an investment only if its rate of return is at least equal, or above the rate of return required by the market for a class of risk corresponding to the riskiness of the firm. In our approach, we are internally setting the rate of return of a new contract by adding all the adjustments that make its rate of return equal to the appropriate rate of return. The latter is determined by the current composition of the assets and their related risks, and by the debt and equity capital financing it, whose costs mirror the leverage and the risk premium above the risk-free rate requested by the debt-holders and shareholders.

In our opinion, in Modigliani and Miller [9] it is Proposition III that has a normative value and that should be considered when designing a framework to evaluate new contracts. Proposition I and II, in the same article, have a positive value describing the equilibrium that can be retrieved *ex post*, equating the return of the assets to the average cost of capital, whichever mix the firm chooses to finance them. But both propositions are not including investments that produce a loss of wealth of one of the stakeholders in favour of another stakeholder: these investments are clearly excluded by Proposition III. Following the latter, we were able to derive the rules that determine the hurdle rate at which the actual contribution of contract to the (shareholders’ bank value) is nil. It is clear that accepting only the investments that comply with Proposition III, also Proposition I and II will be proved to be true, provided we are working in a frictionless, perfect financial market.

5. Conclusion

In this work we have extended the approach of Castagna [5] to a non-trivial setting to calculate the incremental value of a contract that is included in the bank’s balance sheet. A similar approach has been recently developed by see Andersen *et al.* [1]. To our knowledge, our framework is richer than those appeared since now in literature, in that we include a firm structural framework within a classical general equilibrium framework.

The framework considers different financing policies and consistently derives all the adjustment to the “pure” value of a contract, including the **CVA**, the **FVA** and, implicitly, the **LLVA**. We are also able to derive, in a natural fashion, an adjustment that relates to the **KVA**. Although many authors have analysed this type adjustment, a consistent justification of its inclusion in the evaluation process was never provided. For example, the **KVA** is not considered in Andersen *et al.* [1]: the authors admit that “In practice, KVAs are not based on any sort of coherent valuation model”, which is a correct statement given the structural framework they work in.

In our structural, general-equilibrium enhanced framework setting, we do not only flesh out the origin of the **KVA**, but we can also identify the cases in which its inclusion is admissible in the evaluation, which is the correct premium to consider and, moreover, we can spot potential double counting of the adjustment.

One final consideration is about the computational burden that the incremental evaluation requires: it is clear that a Montecarlo engine simulating the balance sheet of the bank in many thousands of scenarios is a necessity. If the impact on the bank's balance sheet of the new contract is negligible, the evaluator may rely on the approximate evaluation, which in practice is the same as the stand-alone pricing, with the relevant xVAs included. In any case, at least for contracts with a stronger impact on the bank's probability of default, and for periodic full revaluations, a robust and sound simulation engine is an indispensable tool.

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